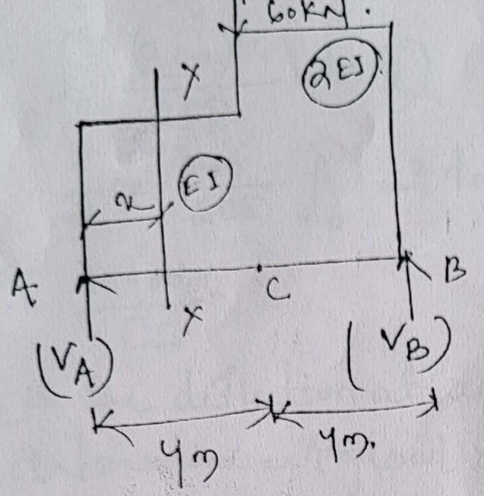
$\Delta = \frac{\partial U}{\partial P}$, $\partial U = \int \Delta \cdot \partial P$

* Strain energy method:-

Using strain energy method determine the deflection under 60KN Load in the beam as shown in figure.

$\Delta = \frac{Pl}{AE}$

$U = \frac{P^2 l}{2AE}$



Solution

Σ upward forces = Σ downward forces

$\Rightarrow V_A + V_B = 60$ —— (1)

$\Sigma M_A = 0.$

$\Rightarrow V_A \times 0 - V_B \times 8 + 60 \times 4 = 0$

$\Rightarrow \boxed{V_B = 30 KN}$ $\quad V_A + \cancel{V_B}^{30} = 60$

$\boxed{V_A = 30 KN}$

Consider a section X-X at a distance x from A.

Moment about section X-X

$M_x = 30 \times x = 30x$

Strain energy stored in a beam $W_1 = \int_0^L \frac{M^2}{2EI} dx$

(3)

$$= \int_0^4 \frac{(30x)^2}{2EI} dx + \int_0^4 \frac{(30x)^2 dx}{2(2EI)}$$

$$= \int_0^4 \frac{(30x)^2}{2EI} \left(1+\frac{1}{2}\right)$$

$$= \frac{3}{2} \times \frac{900}{2EI} \int_0^4 x^2 dx$$

$$= \frac{14400}{EI}$$

y_c is the deflection at centre.

Workdone = Avg Load × deflection

$$= \frac{0+60}{2} \times y_c$$

$$= 30 y_c.$$

$$\Rightarrow 30 y_c = \frac{14400}{EI}$$

$$\Rightarrow \boxed{y_c = \frac{480}{EI}}$$

Comparison between force and displacement method

Force Method:- (FFC method)

In this method, the redundant forces are chosen as unknowns

→ It is called FFC (Force, flexibility and compatibility condition).

→ Force method

→ Flexibility Matrix used

→ Compatibility condition used.

Que 3.6. Determine the vertical deflection at point C in the frame shown in Fig. 3.6.1. Given $E = 200$ kN/mm^2 and $I = 30 \times 10^6$ mm^4.

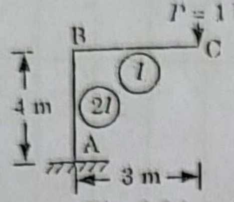


Fig. 3.6.1.

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Answer

Given : Load, $P = 1$ kN, $E = 200$ kN/mm^2, $I = 30 \times 10^6$ mm^4.

To Find : Vertical deflection at point C.

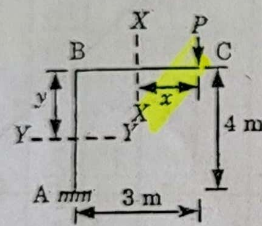

Fig. 3.6.2.

1. Flexural rigidity,

$$EI = 200 \times 30 \times 10^6 \text{ kN-mm}^2 = 6 \times 10^3 \text{ kN-m}^2$$

2. **Expression for Moment :**

Portion	CB	BA
Origin	C	B
Limit	0 – 3	0 – 4
Moment (M)	Px	$3P$

3. Strain energy stored by the frame,

W_i = Strain energy stored by CB + Strain energy stored by BA

$$W_i = \int_C^B \frac{M^2}{2EI} ds + \int_B^A \frac{M^2}{2EI} ds$$

$$W_i = \int_0^3 \frac{(Px)^2}{2EI} dx + \int_0^4 \frac{(3P)^2}{2(2EI)} dy$$

$$= \int_0^3 \frac{P^2x^2}{2EI} dx + \int_0^4 \frac{9P^2}{4EI} dy$$

4. To find the vertical deflection at point C, differentiate the total strain energy stored with respect to P, we get

$$\Delta_c = \frac{\partial U}{\partial P} = \int_0^3 \frac{2Px^2}{2EI} dx + \int_0^4 \frac{9 \times 2P}{4EI} dy$$

$$\Delta_c = \frac{P}{EI}\left[\left(\frac{x^3}{3}\right)_0^3 + \frac{9}{2} \times (y)_0^4\right] = \frac{P}{EI}[9 + 18] = \frac{27 \times 1}{6 \times 10^3}$$

$$\Delta_c = 4.5 \times 10^{-3} \text{ m} = 4.5 \text{ mm} \qquad [\because P = 1 \text{ kN}]$$

Que 3.7. Determine the horizontal displacement of the roller end D of the portal frame shown in Fig. 3.7.1. EI is 8000 kN-m^2 throughout.

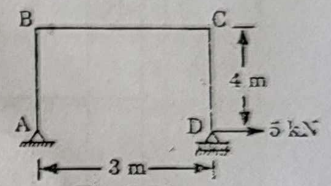

Fig. 3.7.1.

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Answer

Given : Load = 5 kN, EI = 8000 kN-m^2

To Find : Horizontal displacement at D.

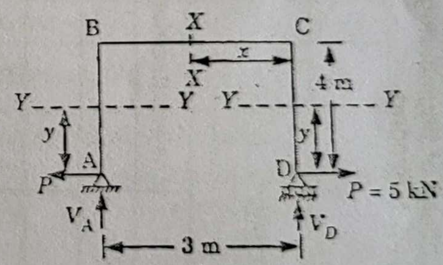

Fig. 3.7.2.

1. Horizontal reaction at $A = P$ (←)
2. Taking moment about point A,

$$V_D \times 3 - 5 \times 0 = 0$$
$$V_D = 0$$

3. Taking moment about support D,

$$V_A \times 3 - 5 \times 0 = 0$$

$V_A' = 0$

4. **Expression for Moments :**

Portion	AB	BC	CD
Origin	A	B	D
Limit	0 – 4	0 – 3	0 – 4
Moment, (M)	Py	4P	Py

5. Strain energy stored by the frame,

$$U_i = \sum \int \frac{M^2}{2EI} ds$$

$$U_i = \int_D^C \frac{M^2}{2EI} ds + \int_C^B \frac{M^2}{2EI} ds + \int_B^A \frac{M^2}{2EI} ds$$

$$U_i = \int_0^4 \frac{(Py)^2}{2EI} dy + \int_0^3 \frac{(4P)^2}{2EI} dx + \int_0^4 \frac{(Py)^2}{2EI} dy$$

$$U_i = \int_0^4 \frac{P^2y^2}{2EI} dy + \int_0^3 \frac{16P^2}{2EI} dx + \int_0^4 \frac{P^2y^2}{2EI} dy$$

6. Horizontal displacement at D is given by,

$$\Delta_D = \frac{\partial U}{\partial P} = \int_0^4 \frac{2Py^2}{2EI} dy + \int_0^3 \frac{32P}{2EI} dx + \int_0^4 \frac{2Py^2}{2EI} dy$$

7. Putting $P = 5$,

$$\Delta_D = \int_0^4 \frac{5y^2}{EI} dy + \int_0^3 \frac{80}{EI} dx + \int_0^4 \frac{5y^2}{EI} dy$$

$$\Delta_D = \frac{5}{EI}\frac{[y^3]_0^4}{3} + \frac{80}{EI}[x]_0^3 + \frac{5}{EI}\frac{[y^3]_0^4}{3}$$

$$= \frac{1}{EI}\left[\frac{5}{3} \times (4^3 - 0) + 80\,(3 - 0) + \frac{5}{3}(4^3 - 0)\right]$$

$$= \frac{1}{EI}\left[\frac{320}{3} + \frac{240}{1} + \frac{320}{3}\right] = \frac{1}{EI}\left[\frac{320 + 720 + 320}{3}\right]$$

Horizontal displacement at D,

$$\Delta_D = \frac{1360}{3 \times 8000} = 0.0567 \text{ m} \approx 57 \text{ mm}$$

Que 3.8. Determine the vertical deflection at the free end and rotation at A in the overhanging beam shown in Fig. 3.8.1. Assume constant EI. Use Castigliano's method.

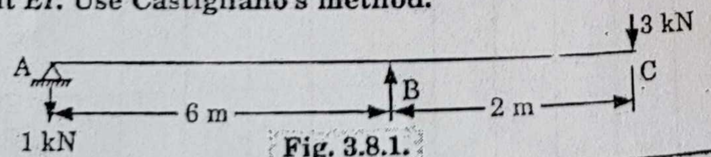

Fig. 3.8.1.

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Answer

Given : Loads = 3 kN and 1 kN, EI = Constant

To Find : Vertical deflection at C and rotation at A.

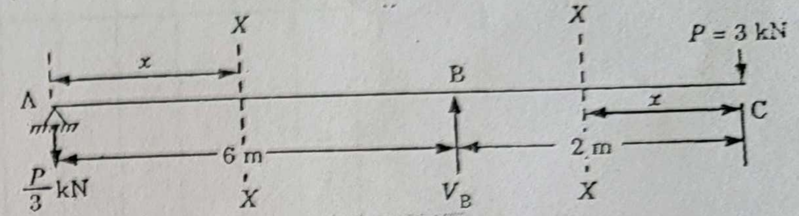


Fig. 3.8.2.

Let load P applied at end point C.

A. Vertical Deflection at Free End :

$$V_B = 1 + 3 = 4 \text{ kN}$$

1. Taking moments about A, $\Sigma M_A = 0$

$$V_B = \frac{4P}{3} (\uparrow)$$

2. $$\Sigma F_y = 0$$

$$V_A' + V_B = 3 + 1 = 4$$

$$V_A = P - \frac{4P}{3} = -\frac{P}{3}$$

3. **Expression for Moment :**

Portion	AB	CB
Origin	A	C
Limit	0 – 6	0 – 2
Moment, (M)	$-\frac{Px}{3}$	Px

4. Strain energy stored by the beam

= Strain energy stored by AB + Strain energy stored by BC

$$U_i = \sum \int \frac{M^2 dx}{2EI} = \int_A^B \frac{M^2}{2EI} dx + \int_C^B \frac{M^2}{2EI} dx$$

$$U_i = \int_0^6 \left(\frac{-P}{3}x\right)^2 \frac{dx}{2EI} + \int_0^2 \frac{(-Px)^2}{2EI} dx$$

$$U_i = \int_0^6 \frac{P^2}{18} x^2 \frac{dx}{EI} + \int_0^2 \frac{P^2x^2}{2EI} dx$$

5. Vertical deflection at free end,

Unit Load Method

Castigliano's second theorem,

$$\delta_i = \frac{\partial U}{\partial P_i}$$

Where,

δ_i is the displacement in the direction of P_i;

U = strain energy stored in the system due to loading

For a beam,

$$U = \int \frac{M^2 dx}{2EI}$$

$$\delta_i = \frac{\partial U}{\partial P_i}$$

$$= \frac{\partial}{\partial P_i} \int \frac{M^2 dx}{2EI}$$

$$= \int \frac{M}{EI} \frac{\partial M}{\partial P_i} dx$$

The partial derivative $\frac{\partial M}{\partial P_i}$ represents the value of the bending moment M caused by a unit load of the load P_i. Thus this derivative is equal to m, which is the bending moment in the structure due to a unit load corresponding to the desired displacement hence,

$$\delta_i = \int \frac{Mm dx}{EI}$$

This is the equation of unit load method when only flexural deformations are considered.

Note: Unit load is applied at the point where slope & deflection is to be found.

Que 3.10. Determine the deflection and rotation at the free end of the cantilever beam shown in Fig. 3.10.1. Use unit load method. Given $E = 2 \times 10^5$ N/mm², and $I = 12 \times 10^6$ mm⁴.

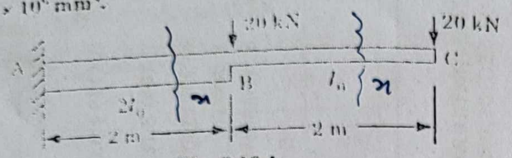
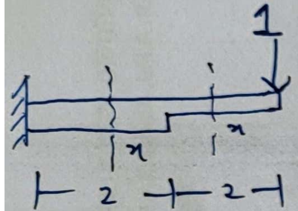


Fig. 3.10.1.



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Answer

Given : Load = 20 kN and 20 kN

$E = 2 \times 10^5$ N/mm² and $I = 12 \times 10^6$ mm⁴

To Find : Deflection and rotation at the free end.

A. Deflection at Point C :

1. Considering the beam into two segments BC and BA and with origin at C and B respectively and measuring x as positive towards left, then the expressions for BM due to external loading and due to unit load applied at B is shown in following table.

Portion	CB	BA
Origin	C	B
Limit	0 – 2	0 – 2
BM due to External Load, M	$20x$	$20(2+x) + 20x = 40 + 40x$
BM due to Unit Load, M_1	$1x = x$	$1(2+x) = 2 + x$

2. Deflection at C,
$$\Delta_C = \int_C^A \frac{MM_1}{EI}dx = \int_C^B \frac{MM_1}{EI}dx + \int_B^A \frac{MM_1}{2EI}dx$$
$$= \int_0^2 \frac{20x(x)}{EI}dx + \int_0^2 \frac{(40+40x)(2+x)}{2EI}dx$$
$$= 20\int_0^2 \frac{x^2}{EI}dx + \int_0^2 \frac{(80+120x+40x^2)}{2EI}dx$$
$$= \frac{20}{EI}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{2EI}\left[80x + \frac{120x^2}{2} + \frac{40x^3}{3}\right]_0^2$$
$$= \frac{20}{EI}\left[\frac{8}{3}\right] + \frac{1}{2EI}\left[80 \times 2 + 60 \times 4 + \frac{40}{3} \times 8\right]$$

$$\Delta = \int_0^L \frac{Mu}{EI}dx$$
$$\theta = \int_0^L \frac{Mu}{EI}dx$$

$$= \frac{53.33}{EI} + \frac{253.33}{EI} = \frac{306.66}{24 \times 10^2} = 0.1277 \text{ m}$$
$$\Delta_C = 127.7 \text{ mm}$$

B. Slope at C :

1. To the find slope at C, apply a clockwise unit moment at B then the parameters are shown in following table.

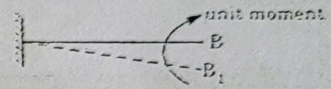

Fig. 3.10.2.

Portion	CB	BA
Origin	C	B
Limit	0 – 2	0 – 2
BM due to External Load, M	$20x$	$20(2+x) + 20x = 40 + 40x$
BM due to Unit Moment, M_1	1	1

2. Slope at C,
$$\theta_C = \int_C^A \frac{MM_1}{EI}dx = \int_C^B \frac{MM_1}{EI}dx + \int_B^A \frac{MM_1}{2EI}dx$$
$$\theta_C = \int_0^2 \frac{20x(1)}{EI}dx + \int_0^2 \frac{(40+40x)(1)}{2EI}dx$$
$$= \int_0^2 \frac{20x}{EI} + \int_0^2 \frac{(40+40x)}{2EI}dx$$
$$= \frac{20}{2EI}[x^2]_0^2 + \frac{1}{2EI}\left[40x + \frac{40x^2}{2}\right]_0^2$$

Slope at C,
$$\theta_C = \frac{10}{EI}(4) + \frac{20}{EI}[2+2] = \frac{120}{24 \times 10^2} = 0.05 \text{ radian}$$

Que 3.11. Determine the horizontal and vertical deflection at point E of the frame shown in Fig. 3.11.1. Take EI as constant.

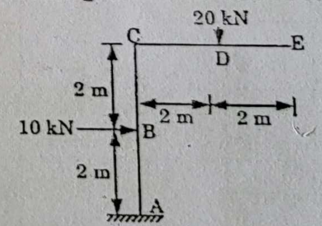

Fig. 3.11.1.

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Answer

Given : Loads = 20 kN and 10 kN

To Find : Horizontal and vertical deflection at point E.

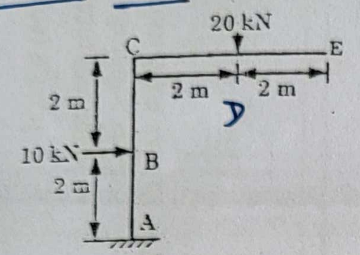

Fig. 3.11.2.

1. **Expression for Moments :**
   Considering sagging moment as positive.

Portion	ED	DC	CB	BA	Remark
Origin	E	D	C	B	
Limit	0 – 2	0 – 2	0 – 2	0 – 2	
M	0	$-20x$	-20×2	$-40 - 10x$ $= -40 - 10x$	BM due to external loads
M_1	$-1 \times x$	$-(2 + x) \times 1$	-4	-4	BM due to unit vertical load at E
M_2	0	0	$-1 \times x$	$-1(2 + x)$	BM due to unit horizontal load at E

2. **Vertical Deflection at E :**

$$\Delta_E = \int_E^A \frac{MM_1 dx}{EI} = \int_E^D \frac{MM_1 dx}{EI} + \int_D^C \frac{MM_1 dx}{EI} + \int_C^B \frac{MM_1 dx}{EI} + \int_B^A \frac{MM_1 dx}{EI}$$

$$= \int 0 - \int_0^2 20x(2 - x) dx + \int_0^2 160 dx + \int_0^2 (160 + 40x) dx$$

$$= \int_0^2 (40x - 20x^2) dx + \int_0^2 160 dx + \int_0^2 (160 + 40x) dx$$

$$= \int_0^2 (20x^2 + 80x + 320) dx$$

$$= \left[20 \frac{x^3}{3} + 80 \times \frac{x^2}{2} + 320 \times x \right]_0^2 = \left[\frac{20 \times 2^3}{3} + \frac{80 \times 2^2}{2} + 320 \times 2 \right]$$

Vertical deflection, $\Delta_E = \dfrac{853.33}{EI}$

3. **Horizontal Deflection at E :**

$$\Delta_{EH} = \int_E^A \frac{MM_2}{EI} dx$$

$$= \int_E^D \frac{MM_2}{EI} dx + \int_D^C \frac{MM_2 dx}{EI} + \int_C^B \frac{MM_2 dx}{EI} + \int_B^A \frac{MM_2 dx}{EI}$$

$$= 0 + 0 + \int_0^2 40x \times dx + \int_0^2 (40 + 10x)(x + 2) dx$$

$$= \int_0^2 (40x) dx + \int_0^2 (40x + 10x^2 + 80 + 20x) dx$$

$$= \int_0^2 (10x^2 + 100x + 80) dx = \left[\frac{10x^3}{3} + \frac{100x^2}{2} + 80x \right]_0^2$$

$$= \left[\frac{10 \times 2^3}{3} + \frac{100 \times 2^2}{2} + 80 \times 2 \right]$$

Horizontal deflection, $\Delta_E = \dfrac{386.67}{EI}$

Que 3.12. **Fig. 3.12.1 shows a pin-joined truss loaded with a single load W is 100 kN. If the area of cross-section of all members shown in Fig. 3.12.1 is 1000 mm², what is the vertical deflection of point C ? Take E = 200 kN/mm² for all members.**

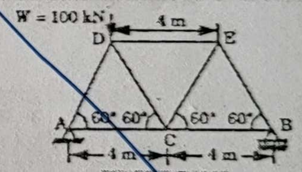

Fig. 3.12.1.

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Answer

Given : Load, W = 100 kN, Cross section area = 1000 mm²

E = 200 kN/mm²

To Find : Vertical deflection at point C.

1. Taking moment about point A,

$$8 \times V_B = 100 \times 2$$

$$= \int_0^1 \frac{-20x \times (-1)}{EI} dx + \int_0^1 \frac{-20(1+x) \times (-1)}{EI} dx$$

$$= \frac{1}{EI}\left(\frac{20x^2}{2}\right)_0^1 + \frac{1}{EI}\left[20x + \frac{20x^2}{2}\right]_0^1$$

$$= \frac{1}{EI}\left[20 \times 1 + 20 \times 1^2\right] = \frac{40}{EI}$$

Slope at B, $\quad \theta_B = \dfrac{40 \times 10^3}{30 \times 10^9 \times 66.67 \times 10^{-6}} = 0.02$ radian

PART-4

Calculation of Deflection by Conjugate Beam Method for Statically Determinate Beam, Trusses and Frame.

CONCEPT OUTLINE : PART-4

Conjugate Beam : Conjugate beam is an imaginary beam of same span as the original beam loaded with $\left(\frac{M}{EI}\right)$ diagram of the original beam, such that the shear force and bending moment at a section will represent the rotation and deflection at the section in the original beam.

Theorem of Conjugate Beam Method :

Theorem 1 : The slope at any section of the given beam is equal to the shear force at the corresponding section of the conjugate beam.

Theorem 2 : The deflection at any section of the given beam is equal to the bending moment at the corresponding section of the conjugate beam.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.14. **Explain conjugate beam theorems.**

Answer

A. **Conjugate Beam Theorems :**

1. These theorems can be derived from moment area theorems and are very useful in finding deflection even if there is no point in the beam where the slope is zero.

2. Now, consider the simple supported beam shown in Fig. 3.14.1 and $\frac{M}{EI}$ diagram in Fig. 3.14.2 respectively.

Now, $\theta_C = \theta_A -$ Area of $\left(\frac{M}{EI}\right)$ diagram between A and C.

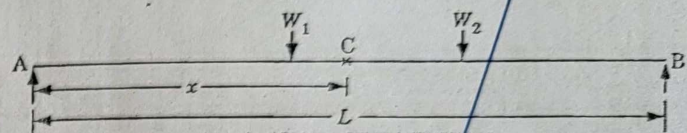

Fig. 3.14.1. A typical beam.

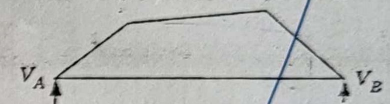

Fig. 3.14.2. *M/EI* diagram.

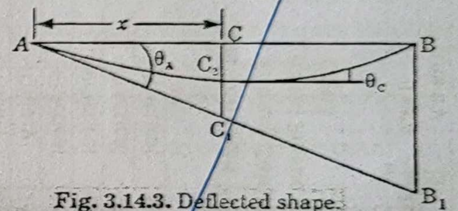

Fig. 3.14.3. Deflected shape.

$$\theta_A = \frac{BB_1}{AB}$$

$$= \left(\frac{1}{L}\right) \text{ moment of area of } \left(\frac{M}{EI}\right) \text{ diagram between } A \text{ and } B \text{ about } B.$$

$$\theta_C = \frac{\text{Moment of } (M/EI) \text{ diagram about } B}{L} - \text{Area of } \left(\frac{M}{EI}\right) \text{ diagram between A and C}$$

3. Now, deflection at C ·

$= CC_2 = CC_1 - C_2C_1 = x_C\,\theta_A -$ Deflection of C w.r.t. tangent at A.

$$= \frac{x_C \times \text{Moment of } (M/EI) \text{ diagram between } A \text{ and } B \text{ about } B}{L} - \text{Area of } \left(\frac{M}{EI}\right) \text{ diagram between } A \text{ and } C \text{ about } C \quad \text{...(3.14.1)}$$

4. Consider an imaginary beam of same span, loaded with $\left(\frac{M}{EI}\right)$ diagram.

5. Then, reaction at A, $V_A = \frac{\text{Moment of the load about } B}{L}$

$$= \frac{\text{Moment of } (M/EI) \text{ diagram between } A \text{ and } B \text{ about } B}{L}$$

6. Shear force at C, = V_A – load between A and C

$$= \frac{\text{Moment of } (M/EI) \text{ diagram between } A \text{ and } B \text{ about } B}{L}$$

$$- \text{Area of } \left(\frac{M}{EI}\right) \text{ diagram between } A \text{ and } C$$

7. Therefore, θ_C in the given beam is equal to the shear force in the beam loaded with $\left(\frac{M}{EI}\right)$ diagram.

8. Similarly, it can be observed that the deflection of C, given by eq. (3.14.1.), is equal to the bending moment in the imaginary beam loaded with $\left(\frac{M}{EI}\right)$ diagram.

8. The imaginary beam is called the conjugate beam and from the above discussion the following two theorems result :

Theorem 1 : The rotation at a point in a beam is equal to the shear force in the conjugate beam.

Theorem 2 : The deflection in a beam is equal to the bending moment in the conjugate beam.

Que 3.15. **Determine the slope and deflection at the free end of a cantilever beam of span L subjected to a point load W at the free end using any method for your choice. Take EI as constant.**

AKTU 2012-13, Marks 10

OR

A cantilever of span L is subjected to a point load W at the free end. Determine the slope and deflection at the free end. Take EI as constant.

AKTU 2013-14, Marks 05

OR

Determine the deflection at free end of a cantilever beam.

AKTU 2014-15, Marks 10

Answer

1. Consider a cantilever beam AB of length L carrying a point load W at free end.

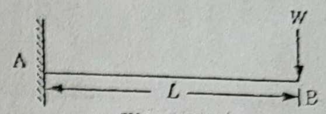

Fig. 3.15.1.

2. Bending moment diagram for cantilever is shown in Fig. 3.15.2.

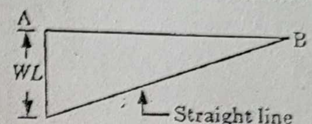

Fig. 3.15.2. BM diagram.

Corresponding conjugate beam whose load diagram is the $\frac{M}{EI}$ diagram.

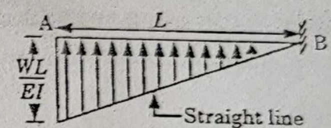

Fig. 3.15.3. Conjugate beam diagram (M/EI diagram).

3. **Slope at Free End in Cantilever Beam :**

Slope at B for the given beam = Shear force at B for conjugate beam.

$$\theta_B = \text{Area of } \frac{M}{EI} \text{ diagram}$$

$$\theta_B = \frac{1}{2} \times \frac{WL}{EI} \times L = \frac{WL^2}{2EI}$$

4. **Deflection at Free End in Cantilever Beam :**

Deflection at B for the given beam = BM at B for conjugate beam.

$$\Delta_B = \frac{1}{2} \times WL \times L \times \frac{2L}{3} = \frac{WL^3}{3}$$

Que 3.16. **Determine the slope at the supports and deflection at the mid span of a simply supported beam AB of span L subjected to a point load W at the mid span. Take EI as constant. Use any method of your choice.**

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Answer

Given : Concentrated load = W, Span of beam = L.

To Find : Deflection at mid span and slope at supports.

1. Fig. 3.16.1(a) shows the real beam; BM diagram for the real beam is shown in the Fig. 3.16.1(b).

2. The $\frac{M}{EI}$ diagram of the real beam becomes the elastic weight or loading for the conjugate beam. The conjugate beam $A'C'B'$ (corresponding to the real beam ACB) with the loading is shown in Fig. 3.16.1(c).

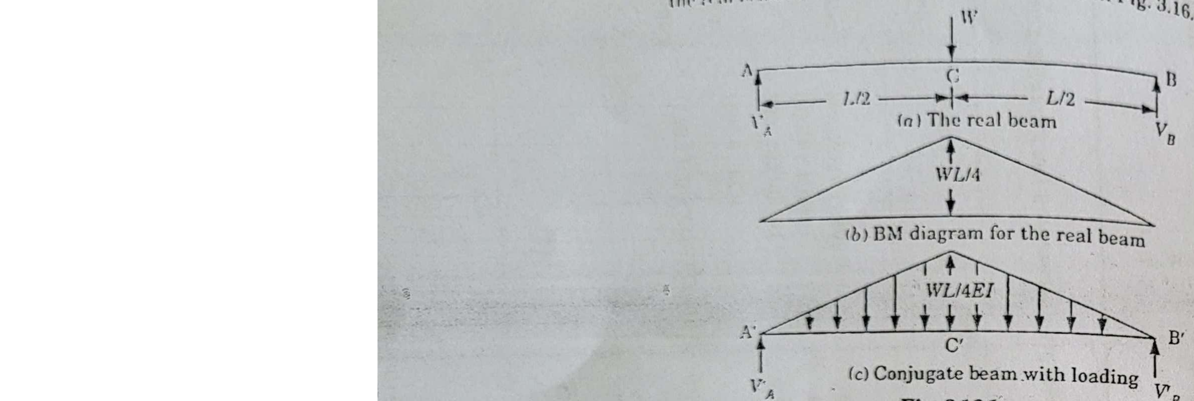

Fig. 3.16.1.

3. For the conjugate beam :

$$V'_A = \frac{1}{2}\left(\frac{1}{2}L \times \frac{WL}{4EI}\right) = \frac{WL^2}{16EI}$$

4. But shear at any section of the conjugate beam is equal to the slope of the real beam.

i. Hence, θ_A = Slope at the end A of real beam $= -\frac{WL^2}{16EI}$

ii. Similarly, $\theta_B = S'_B = V'_B = \frac{WL^2}{16EI}$

5. Again, for the conjugate beam,

$$M'_C = V'_A \times \frac{L}{2} - \left(\frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4EI}\right) \times \left(\frac{1}{3} \times \frac{L}{2}\right)$$

$$= \frac{WL^2}{16EI} \times \frac{L}{2} - \frac{WL^3}{96EI} = \frac{WL^3}{48EI}$$

6. But the BM at any section of the conjugate beam is equal to the deflection of the real beam.

Hence, $\Delta_C = M'_C = \frac{WL^3}{48EI}$