Solⁿ 10:-

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$$

Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\, dx}{1+\cos^2(\pi - x)} \quad \text{(by } P_4\text{)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x\, dx}{1+\cos^2 x} = \pi \int_0^{\pi} \frac{\sin x\, dx}{1+\cos^2 x} - I$$

or $2I = \pi \int_0^{\pi} \frac{\sin x\, dx}{1+\cos^2 x}$

or $I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x\, dx}{1+\cos^2 x}$

Put $t = \cos x$ so that $-\sin x\, dx = dt$, when $x = 0$, $t = 1$ and $x = \pi$, $t = -1$. Therefore, (by P_1) we get

$$I = \frac{-\pi}{2} \int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2} \int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi \int_0^1 \frac{dt}{1+t^2} \quad \left(\text{by } P_7 \text{, since } \frac{1}{1+t^2} \text{ is even function}\right)$$

$$= \pi \left[\tan^{-1} t\right]_0^1 = \pi \left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= \pi \left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi^2}{4} \quad \text{Ans}$$