$$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$$

$$\Rightarrow \int_0^{\pi} \frac{(\pi - x)\ \sin(\pi - x)\ dx}{1+\cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x) \sin x}{1+\cos^2 x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x\ dx}{1+\cos^2 x} - I$$

$$2I = \pi \int_0^{\pi} \frac{\sin x\, dx}{1+\cos^2 x}$$

$$I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x\ dx}{1+\cos^2 x}$$

$\cos x = t$

$-\sin x\ dx = dt$

$x = 0,\ t = 1$ and $x = \pi,\ t = -1,$

$$I = -\frac{\pi}{2}\int_{1}^{-1} \frac{dt}{1+t^2}$$

$$= \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi\int_{0}^{1} \frac{dt}{1+t^2}$$

$$= \pi\left[\tan^{-1} t\right]_0^1$$

$$= \pi\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= \pi\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi^2}{4}.$$