Sec - 10

2 Ans $y'' + (3\sin x - \cot x) y' + 2y\sin^2 x = e^{-\cos x}$

Changing independent variable $z = f(x)$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}, \quad \frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dx} + \frac{d^2z}{dx^2}$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)\left(\frac{dz}{dx}\right) + \frac{d^2z}{dx^2}$$

$$= \frac{d^2y}{dz^2}\left(\frac{dz^2}{dx}\right) + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Now from given equation.

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{d^2z}{dx^2} + (3\sin x - \cot x)\frac{dy}{dz}\frac{dz}{dx} + 2y\sin^2 x = e^{-\cos x}$$

$$\frac{d^2y}{dz^2} + \frac{d^2z}{dx^2} + \frac{(3\sin x - \cot x)\frac{dy}{dy}\frac{dz}{dx}}{\frac{dz^2}{dx}} + \frac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2}y = e^{\cos x}$$

This can be rewritten as $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

where $P_1 = \dfrac{\frac{d^2y}{dz^2} + (3\sin x - \cot x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$

$$Q_1 = \frac{2\sin^2 x}{(dz/dx)^2}, \quad R_1 = \frac{e^{-\cos x}\sin^2 x}{(dz/dx)^2}$$

Choose $Q_1 = 2i.e., z = \frac{2\sin^2 x}{(dz/dx)^2} \Rightarrow \left(\frac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \frac{dz}{dx} = \sin x$

$z = -\cos x$

$\frac{d^2z}{dx^2} = \cos x$

Now: $P_1 = \frac{\cos x + (3\sin x - \cot x)\sin x}{\sin^2 x}$

$= \frac{\cos x + 3\sin^2 x - \frac{\cos x}{\sin x}\sin x}{\sin^2 x} = 3$

$R_1 = \frac{e^{-\cos x}\sin^2 x}{\sin^2 x} = e^{-\cos x}$

$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{-\cos x}$

$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{-z}$

Auxilliary equation is $m^2 + 3m + 2 = 0$

$m = -1, -2$

$C.F. = C_1 e^{-z} + C_2 e^{-2z}$

$P.I. = \frac{1}{(D+2)(D+1)} e^{z} = \frac{1}{D^2+3D+2} e^{z}$

$D = -1$

$\frac{1}{1+3+2} e^{z} = \frac{e^{z}}{6}$