Section. 08

Q-1

Ans: Given as - Ellipsoid, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —(1)

at the point (x,y,z)

Density $\boxed{\rho = Kxyz}$

We know that

$\boxed{\text{Mass} = \text{volume} \times \text{Density}}$

So volume of the solid bounded by the ellipsoid

$v = \iiint_A dx\,dy\,dz$

Let $\frac{x^2}{a^2} = u \Rightarrow 2x\,dx = a^2 du$, $\frac{y^2}{b^2} = v$, $\frac{z^2}{c^2} = w$ —(1)

$\Rightarrow dx = \frac{a^2}{2x}du = \frac{a^2}{2\times a\sqrt{u}}du$

$\Rightarrow \boxed{dx = \frac{a}{2\sqrt{u}}du}$

Similarly

$\boxed{dy = \frac{b}{2\sqrt{v}}dv}$

$\boxed{dz = \frac{c}{2\sqrt{w}}dw}$

So required volume $V = \iiint_\theta \frac{a}{2\sqrt{u}} \times \frac{b}{2\sqrt{v}} \times \frac{c}{2\sqrt{w}}\,du\,dv\,dw$

$\Rightarrow V = \iiint_\theta \frac{abc}{8\times\sqrt{uvw}}\,du\,dv\,dw$

when θ is the region when $u\geq 0$, $v\geq 0$, $w\geq 0$ and $u+v+w=1$

$\Rightarrow V = \frac{abc}{8}\iiint_\theta u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw$

By using Dirichlet's integral

$\Rightarrow \iiint_\theta x^{a-1}y^{b-1}z^{c-1}\,dx\,dy\,dz = \frac{\sqrt{a}\sqrt{b}\sqrt{c}}{\sqrt{a+b+c+1}}$

$\Rightarrow V = \frac{abc}{8}\iiint u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw$

$\Rightarrow V = \frac{abc}{8}\times\frac{\sqrt{1/2}\sqrt{1/2}\sqrt{1/2}}{\sqrt{1/2+1/2+1/2+1}}$

$V = \frac{abc}{8}\times\frac{(\sqrt{\pi}\times\sqrt{\pi}\times\sqrt{\pi})}{\sqrt{3/2+1}}$

$\Rightarrow V = \frac{abc}{8}\times\frac{\pi\sqrt{\pi}}{\sqrt{5/2}}$

$\Rightarrow V = \frac{abc}{8}\times\frac{\pi\sqrt{\pi}}{3/2\times 1/2\sqrt{1/2}} = \frac{abc}{8}\times\frac{\pi\sqrt{\pi}}{\frac{3}{4}\sqrt{\pi}}$

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$$\Rightarrow \left[V = \frac{abc}{8} \times \frac{4\pi}{3} = \frac{\pi}{6} abc \right] \text{ cubic units}$$

$\Rightarrow$ Mass = Volume × Density

$$\Rightarrow M = \iiint_{\theta} K\, xyz \, dx\, dy\, dz$$

$$\because \frac{x^2}{a^2} = u \Rightarrow \boxed{x = a\sqrt{u}} \Rightarrow dx = \frac{a}{2\sqrt{u}} du$$

$$\frac{y^2}{b^2} = v \Rightarrow \boxed{y = b\sqrt{v}} \Rightarrow dy = \frac{b}{2\sqrt{v}} dv$$

$$\frac{z^2}{c^2} = w \Rightarrow \boxed{z = c\sqrt{w}} \Rightarrow dz = \frac{c}{2\sqrt{w}} dw$$

Put the value in above eqn

$$M = \iiint_{\theta} K \times a\sqrt{u} \times b\sqrt{v} \times c\sqrt{w} \times \frac{a\, du}{2\sqrt{u}} \times \frac{b\, dv}{2\sqrt{v}} \times \frac{c\, dw}{2\sqrt{w}}$$

$$\Rightarrow M = \iiint_{\theta} K \frac{a^2b^2c^2}{8} du\, dv\, dw = \frac{Ka^2b^2c^2}{8}$$

$$\iiint_{\theta} u^2 v^2 w^2 \, dv\, du\, dw$$

$$M = \frac{Ka^2b^2c^2}{8} \iiint_{\theta} u^{1-1} v^{1-1} w^{1-1} du\, dv\, dw$$

Again using Dirichlet's Integral

$$\Rightarrow M = \frac{Ka^2b^2c^2}{8} \times \frac{\sqrt{1}\sqrt{1}\sqrt{1}}{\sqrt{1+1+1+1}}$$

$$M = K\frac{a^2b^2c^2}{8} \times \frac{1 \times 1 \times 1}{\sqrt{4}}$$

$$M = \frac{Ka^2b^2c^2}{8} \times \frac{1}{3 \times 2 \times 1}$$

$$\left[M = \frac{Ka^2b^2c^2}{48} \right] \text{Ans}$$

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