Section - 3

Ans - No - 1.

Solution ① We have

$$I = \int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta} d\theta \quad — ⓒ$$

∵ we know that

$e^{i\theta} = \cos\theta + i\sin\theta$ — ①

$e^{-i\theta} = \cos\theta - i\sin\theta$ — ⓘⓘ

from eqn ① & ⓘⓘ

$\Rightarrow \cos = e^{i\theta} + e^{-i\theta}$

$$\boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

Let $e^{i\theta} = z$

Diff both side w.r to θ

$\Rightarrow dz = ie^{i\theta} d\theta = iz d\theta$

$$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$$

Putting above value in eqn ⓒ we get

$$I = \oint_c \frac{(z-z^{-1})}{3+\frac{1}{2i}(z-z^{-1})} \times \frac{dz}{iz}$$

$$I = \oint_c \frac{(z-\frac{1}{z})}{3+\frac{1}{2i}(z-\frac{1}{z})} \times \frac{dz}{iz}$$

$$I = \oint_c \frac{z-\frac{1}{z}}{3+\frac{1}{2i}(z-\frac{1}{z})} \frac{dz}{iz}$$

$$I = \oint_c \left(\frac{\frac{z^2-1}{z}}{3+\frac{z^2-1}{2iz}}\right) \times \frac{dz}{iz}$$

$$\Rightarrow I = \oint_c \frac{z^2-1}{z\left(\frac{6iz+z^2-1}{2\cancel{i}z}\right)} \times \frac{dz}{\cancel{iz}}$$

$$\Rightarrow I = 2\oint_c \frac{(z^2-1)}{z(z^2+6iz-1)} dz$$

$\Rightarrow I = 2\oint_c f(z)\, dz$,

where c is unit circle $(|z|=1)$

The poles are $\boxed{z=0}$

$\Rightarrow z^2+6iz-1=0 \Rightarrow z = \frac{(-6i) \pm \sqrt{36i^2+4}}{2}$

$$z = \frac{-6i \pm \sqrt{-36+4}}{2} = \frac{-6i \pm \sqrt{-32}}{2}$$

~~$z = -6i \pm 4\sqrt{2}i$~~

$z = -3i + 2\sqrt{2}i$

$\boxed{z = -3i + 2.828i}$

$z = -3i - 2\sqrt{2}i$

$z = -3i - 2.828i$

$z = \boxed{-5.828i}$

The poles are $z=0$, $z=-0.2i$ and $z=-5.83i$
of these only $z=0$ and $z=-0.2i$ lies within
in the circle

$\therefore \; Res\, f(0) = \lim_{z\to a} (z-a) f(z)$

$\Rightarrow Res\, f(0) = \lim_{z\to 0} \left[\dfrac{(z-0)\,(z^2-1)}{z(z^2+6iz-1)} \right]$

$= \lim_{z\to 0} \left[\dfrac{z\,(z^2-1)}{z(z^2+6iz-1)} \right]$

Put $z \to 0$

$\Rightarrow Res\, f(0) = \dfrac{0-1}{0+0-1} = \dfrac{+1}{+1}$

$\boxed{Res\, f(0) = 1}$

$\Rightarrow Res\, f(-0.2i) = f\left(-\frac{1}{5}i\right) = \lim_{z\to -\frac{1}{3}i} \left[\dfrac{\left(z+\frac{1}{3}i\right)(z^2-1)}{z\left(z+\frac{1}{5}i\right)(z+3i+2\sqrt{2}i} \right]$

$\Rightarrow Res\, f\left(-\frac{1}{5}i\right) = \lim_{z\to -\frac{1}{5}i} \left(\dfrac{z^2-1}{z(z+3i+2\sqrt{2}i)} \right]$

Put $\Rightarrow \frac{1}{3}i$

$\Rightarrow Res\, f\left(-\frac{1}{3}i\right) = \dfrac{\frac{i^2}{25}-1}{\dots}$

$= \dfrac{-1-25}{\frac{i^2}{25} - \frac{3i^2}{5} - \frac{2\sqrt{2}i^2}{5}}$

$\Rightarrow Res\, f\left(-\frac{1}{3}i\right) = \dfrac{\frac{-26}{\cancel{25}}}{\frac{i^2-15i^2-10\sqrt{2}i^2}{\cancel{25}}} = \dfrac{-26}{-1+15+10\sqrt{2}}$

$\boxed{Res\, f\left(-\frac{1}{3}i\right) = \dfrac{-26}{14+10\sqrt{2}}}$

Hence $I = 2\left\{ 2\pi i \left(Res\, f(0) + Res\, f\left(-\frac{1}{3}i\right) \right) \right\}$

$= 4\pi i \left[1 + \dfrac{-26}{14+10\sqrt{2}} \right]$

$= 4\pi i \left[\dfrac{14+10\sqrt{2}-26}{14+10\sqrt{2}} \right]$

$I = 2\pi i \left[\dfrac{10\sqrt{2}-12}{7+5\sqrt{2}} \right]$