Section- 4

2. if $f(z)$ is a harmonic function of z show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2=|f'(z)|^2$$

Ans. → We have $f(z) = u+iv$ —— ①

$\therefore |f(z)| = \sqrt{u^2+v^2}$ —— ②

Partially differentaiting eq ② with respect x and y, we get

$$\frac{\partial}{\partial x}|f(z)| = \frac{1}{2}(u^2+v^2)^{-1/2}\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right)$$

$$= \frac{u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}}{|f(z)|} \quad \text{—— ③}$$

Similarly $\frac{\partial}{\partial y}|f(z)| = \frac{u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}}{|f(z)|}$ —— ④

Squaring and adding eq ③ and ④ we get

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)\right\}^2 = \frac{\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)^2+\left(u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}\right)^2}{|f(z)|^2}$$

$$= \frac{\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)^2+\left(-u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)^2}{|f(z)|^2}$$

Using C-R eqn

$$= \frac{(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]}{|f(z)|^2}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \qquad (\because |f(z)|^2 = u^2+v^2)$$

$$= |f'(z)|^2 \qquad \left(\because f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$