Given as - the function

$$f(x) = \begin{cases} -1 & -\pi < x < -0.5\pi \\ 0 & -0.5\pi < x < 0.5\pi \\ 1 & 0.5\pi < x < \pi \end{cases}$$

The Fourier Series for the function -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (1)$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} (-1)\, dx + \int_{\pi/2}^{\pi} (1)\, dx\right]$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left\{-[x]_{-\pi}^{\pi/2} + [x]_{\pi/2}^{\pi}\right\}$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left\{-\left[-\frac{\pi}{2} + \pi\right] + \left[\pi - \frac{\pi}{4}\right]\right\}$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left\{+\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right\}$$

$$\Rightarrow a_0 = \frac{1}{\pi}(0)$$

$$\Rightarrow \boxed{a_0 = 0}$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos n\pi\, dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} -\cos nx\,dx + \int_{\pi/2}^{\pi} \cos nx\,dx\right\}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left\{-\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2} + \left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}\right\}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left\{-\left[\frac{\sin\left(\frac{-n\pi}{2}\right) - \sin(-n\pi)}{n}\right] + \left[\frac{\sin n\pi - \sin\frac{n\pi}{2}}{n}\right]\right\}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left\{\frac{\sin\frac{n\pi}{2}}{n} - \frac{\sin n\pi}{n} + \frac{\sin n\pi}{n} - \frac{\sin(n\pi/2)}{n}\right\}$$

$$\Rightarrow a_n = \frac{1}{\pi}(0)$$

$$\Rightarrow \boxed{a_n = 0}$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} -\sin nx\,dx + \int_{\pi/2}^{\pi} \sin nx\,dx\right\}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left\{\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}\right\}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left\{\frac{\cos\left(\frac{-n\pi}{2}\right) - \cos(-n\pi)}{n} - \left(\frac{\cos n\pi - \cos\frac{n\pi}{2}}{n}\right)\right\}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left\{\frac{\cos\frac{n\pi}{2} - \cos n\pi}{n} - \left(\frac{\cos n\pi - \cos n\pi/2}{n}\right)\right\}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left\{\frac{\cos\frac{n\pi}{2} - \cos n\pi - \cos n\pi + \cos\frac{n\pi}{2}}{n}\right\}$$

Hence required series is -

$$f(x) = \frac{0}{2} + \sum^{\infty} (0) \cos nx + \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\cos\frac{n\pi}{2} - \cos n\pi \right) \sin nx$$

Putting $x = \frac{\pi}{2}$ in above series -

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{\cos n\pi}{2} - \cos n\pi \right] \sin \frac{n\pi}{2}$$

$$\Rightarrow \frac{0+1}{2} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos\frac{n\pi}{2} - \cos n\pi \right] \sin \frac{n\pi}{2}$$

Put $n = 1, 2, 3, 4 \ldots$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1} \left[\cos\frac{\pi}{2} - \cos\pi \right] \sin\frac{\pi}{2} + \frac{1}{2} \left[\cos\pi - \cos 2\pi \right] \sin\pi + \frac{1}{3} \left[\cos\frac{3\pi}{2} - \cos 3\pi \right] \sin\frac{3\pi}{2} + \frac{1}{4} \left[\cos 2\pi - \cos 4\pi \right] \sin 2\pi + \ldots$$

$$\Rightarrow \frac{\pi}{4} = 1 [0 - (-1)] 1 + \cancel{0} 0 + \frac{1}{3} [(-1)] + 0 + \frac{1}{5}(1) + 0 + \frac{1}{7} (-1) + \ldots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Ans