Derivation of $E = mc^2$

The mass m of a body moving with velocity v relative to a stationary observer varies with v and is given by

$$\left(m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \right)$$

Squaring

$$m^2 = \frac{m_0^2}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$m^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2$$

$$m^2\left(\frac{c^2 - v^2}{c^2}\right) = m_0^2$$

$$m^2c^2 - m^2v^2 = m_0^2 c^2$$

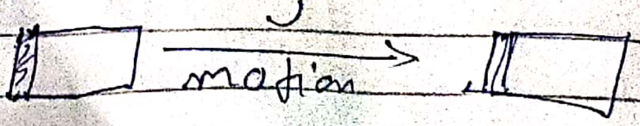
Diff

$$2mc^2\, dm - [2mv^2\, dm + m^2 2v\, dv] = 0$$

$$2mc^2 dm - 2mv^2 dm - 2m^2 v dv = 0$$

$$c^2 dm - v^2 dm - mv dv = 0$$

$$c^2 dm = (v^2 dm + mv dv)$$



$$dk = dw = F\,ds \quad \text{—} (2)$$

m from Newton second law

$$F = \frac{dP}{dt} \qquad P = mv$$

$$F = \frac{d}{dt} mv$$

$$f = m\frac{dv}{dt} + v\frac{dm}{dt}$$

$$dk = \left(m\frac{dv}{dt} + v\frac{dm}{dt}\right) \cdot ds$$

$$dk = m\frac{dv}{dt}ds + v\frac{dm}{dt}ds$$

$$dk = m\frac{ds}{dt}dv + v\frac{ds}{dt}dm$$

$$dk = mv\,dv + v^2 dm \quad -(3)$$

compare (1) and (3)

$$dk = c^2 dm$$

Now Integrate :

$$\int_0^k dk = \int_{m_0}^{m} c^2 dm$$

$$K = c^2 [m]_{m_0}^{m}$$

$$K = c^2 (m - m_0)$$

Total Energy

$$E = K.E + P.E$$

$$= c^2(m - m_0) + m_0$$

$$= mc^2 - \cancel{m_0c^2} + \cancel{m_0c}$$

$$= mc^2$$

Ans