One dimensional wave equaction:-

One dimensional wave equaction is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}$$

Where $c^2 = \frac{T}{m}$ —(1)

T = Tensile (Tension) in the string
m = Mass per unit length of the string

Solution of one dimensional wave equaction is done by method of separation of variables.

Let $u = X(x)T(t)$ —— (2)

Where X is a function of x only and T is a function of t only.

Differentaiting eq. (2) partially w.r.t x and t respectively and putting the values eq. (1)

$$\frac{\partial^2 u}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2} \quad \text{and}$$

$$\frac{\partial^2 u}{\partial x^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

~~separating~~

From one dimensional waave equaction

$$X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

Separating the variables,

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = -k^2 = k^2 \text{ or } 0 \text{ (say)}$$

Case (i):-

When $\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k^2$ and $\frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = -k^2$

Or $\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$ and $\frac{\partial^2 X}{\partial t^2} + k^2 c^2 T = 0$

or $(D^2 + k^2) X = 0$ and $(D^2 + k^2 c^2) T = 0$

Auxiliary equaction are $m^2 + k^2 = 0$ and $m^2 + k^2 c^2 = 0$

$m = \pm ki$ and $m = \pm kci$

Thus complementary functions are

$$X = c_1 \cos kx + c_2 \sin kx$$

and $T = c_3 \cos kct + c_4 \sin kct$

$\therefore \quad u = XT$

$$\Rightarrow \quad u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos kcT + c_4 \sin kcT) \quad (3)$$

Case (ii) :-

When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = k^2$

$m = k^2$ and $m^2 = k^2 c^2$

$m = \pm k$ and $m = \pm kc$

$$\Rightarrow \quad X = c_5 e^{kx} + c_6 e^{-kx}$$

and $T = c_7 e^{kx} + c_8 e^{-kcT}$

Thus $u = (c_5 e^{kx} + c_6 e^{-kx})(c_7 e^{kct} + c_8 e^{-kcT}) - (4)$

Case (iii):- When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = 0$

$m = 0, 0$ and $m, 0 0$

$\Rightarrow X = C_9 + C_{10} x$ and $T = C_{11} + C_{12} t$

Thus $u = (C_9 + C_{10} x)(C_{11} + C_{12} t)$ — (5)

The ~~solutdal~~ solution given by (3) satisfies the one dimensional wave equation.