Section-1

Q3

Ans → One dimensional wave equation is given by :-

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{m} \quad \text{—— (1)}$$

T = Tension in the string.
m = Mass per unit length of the string.

Solution of one Dimensional wave eq^n. is done by method of separation of variable.

Let $u = X(x).T(t)$ —— (2)

where,
X is a function of x &
T is a function of t only.

Differentiating eq. (2) partially w.r.t x & t respectively and putting the value in eq^n. (1)

$$\frac{\partial^2 u}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2} \quad \text{and}$$

$$\frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$$

from one dimensional wave eq^n.

$$\boxed{X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}}$$

separating the variable,

Case (i)

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T}\cdot\frac{\partial^2 T}{\partial t^2} = -k^2 \text{ or } k^2 \text{ or } 0 \text{ (say)}$$

(we use $-k^2$)

or $\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$ $\frac{\partial^2 T}{\partial t^2} + k^2 c^2 T = 0$

or

$$(D^2 + k^2)X = 0 \ \& \ (D^2 + k^2 c^2)T = 0$$

Auxiliary eq^n. term $m^2 + k^2 = 0$ & $m^2 + k^2c^2 = 0$

$$m = \pm ki \ \& \ m = \pm kci$$

thus complementary function are

$$X = c_1 \cos kx + c_2 \sin kx$$

&

$$T = c_3 \cos kct + c_4 \sin kct$$

$\therefore$ $u = XT$

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos kct + c_4 \sin kct) \longrightarrow (3)$$

Case (ii)

When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$ and $\frac{1}{c^2 T}\cdot\frac{\partial^2 T}{\partial t^2} = k^2$

$$m = k^2 \ \& \ m^2 = k^2 c^2$$

$$m \pm k \ \& \ m \pm kc$$

$$\Rightarrow \boxed{X = c_5 e^{kx} + c_6 e^{-kx}}$$

and

$$T = C_7 e^{kct} + C_8 e^{-kct}$$

thus,

$$u = (C_5 e^{kx} + C_6 e^{-kx})(C_7 e^{kct} + C_8 e^{-kct}) \quad \text{--- (4)}$$

Case (iii):

When $\frac{1}{X} \cdot \frac{\partial^2 X}{\partial x^2} = 0$ & $\frac{1}{C^2 T} \cdot \frac{\partial^2 T}{\partial t^2} = 0$

$m = 0, 0$, and $m = 0, 0$

$$\Rightarrow X = C_9 + C_{10} x \text{ and } T = C_{11} + C_{12} t$$

Thus,

$$u = (C_9 + C_{10} x)(C_{11} + C_{12} t) \quad \text{--- (5)}$$

- The solⁿ given eqⁿ ③ satisfies the one-dimensional wave eqⁿ.
  Thus the required solⁿ of one-dimensional wave eqⁿ ③
  Now to find the value of C_1, C_2 & C_3 & C_4 which are obtained by applying boundary & initial conditions.

- Initially if the string is at, $g(x) = 0$. Now to find the constant of eqⁿ. ③ apply the boundary condⁿ (i) to eqⁿ. ③, we get

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$$0 = c_1 (c_3 \cos kct + c_4 \sin kcT)$$

$\Rightarrow$ $c_1 = 0$ $\quad [\because c_3 \cos kct + c_4 \sin kct \neq 0]$

from eqn. (3), we get

$$u = c_2 \sin kx (c_3 \cos kct + c_4 \sin kct) \quad \text{--- (6)}$$

Now put boundary condn. (ii) in eqn. (6), we get

$$0 = c_2 \sin kl (c_3 \cos kct + c_4 \sin kct)$$

$$\sin kl = 0$$
$$kl = n\pi$$
$$\boxed{k = \frac{n\pi}{l}}$$

from eqn. (5), we get

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right)$$

$$u = \sin \frac{n\pi x}{l} \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \quad \text{--- (7)}$$

Now apply initial conditions (iii) & (iv)

$$u = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = f(x) \quad \text{--- (8)}$$

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{t=0}^{\infty} \frac{n\pi C}{l} \cdot B_n \sin \frac{n\pi x}{l} = g(x)$ —— (9)

- The left hand side of the eq^n (8) and eq^n (9) respresents Fourier sine expansion of the right hand side. Thus,

$$A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \cdot dx$$

&

$$\frac{n\pi C}{l} \cdot B_n = \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Putting the value of A_n & B_n in eq^n (7), we obtain the required solution of one-dimensional wave equation. Ans.