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Section-1

Q.1
Ans. Solution:-
Given Laplace equation is :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

Let $u = XY$,
where $X$ is the only function of $x$ &
$Y$ is the only function of $y$.

$$\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2}$$

&

$$\frac{\partial^2 u}{\partial y^2} = X \frac{\partial^2 Y}{\partial y^2}$$

from eq^n (1), we get

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

Case (i)

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \text{ (say)}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$$

and

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$X = c_1 x + c_2$$
$$Y = c_3 y + c_4$$

at

$y = 0, Y = 0 \Rightarrow c_4 = 0$

also, $y = b, Y = 0 \Rightarrow c_3 = 0$

$\therefore$ $Y = 0$

Thus, $u = XY = X(0)$

$u = 0$ (not possible)

Case (ii)

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 \text{ (say)}$$

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$$

~~Ver~~ and

$$\frac{\partial^2 Y}{\partial y^2} - k^2 Y = 0$$

$$X = C_1 \cos kx + C_2 \sin kx,$$
$$Y = C_3 e^{ky} + C_4 e^{-ky}$$

If $y = 0$, $Y = 0$

$$C_3 + C_4 = 0$$
$$\boxed{C_4 = -C_3}$$

and

$$Y = 0 \text{ at } y = b$$

$$0 = C_3 e^{kb} - C_3 e^{-kb}$$

$$C_3(e^{kb} - e^{-kb}) = 0$$
$C_3 = 0$ & $C_4 = 0$ & $Y = 0$ (not possible)

Case (iii)

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (say)}$$

$$X = C_3 \cos ky + C_4 \sin ky$$

at $y = 0$, $Y = 0$, $C_3 = 0$
$$Y = C_4 \sin ky$$

at $y = b, \; Y = 0$

$$0 = C_4 \sin kb$$
$$\sin kb = 0$$
$$kb = n\pi$$
$$\boxed{k = \frac{n\pi}{b}}$$

thus,

$$u = (c_1 e^{kx} + c_2 e^{-kx}).C_4 \sin \frac{n\pi y}{b} \quad \text{--- (2)}$$

at $x = 0, \; u = 0$

$$0 = (c_1 + c_2).C_4 \sin \frac{n\pi y}{b}$$
$$c_1 + c_2 = 0$$
$$\boxed{c_2 = -c_1}$$

from eq$^n$ (2), we get

$$u = \frac{2}{2} c_4 c_1 (e^{kx} - e^{-kx}).\sin \frac{n\pi y}{b}$$
$$u = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}}}{2}\right).\sin\left(\frac{n\pi y}{b}\right) \quad \text{--- (3)}$$

let

$$b_n = 2 c_1 . c_4$$
$$x = a, \; u = f(y)$$

from eq$^n$ 3, we get

$$f(y) = \sum_{n=0}^{\infty} b_n \left( \frac{e^{\frac{n\pi a}{b}} - e^{\frac{-n\pi a}{b}}}{2} \right) \sin\left(\frac{n\pi y}{b}\right)$$

$$f(y) = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi a}{b}\right) \cdot \sin\left(\frac{n\pi y}{b}\right)$$

$$b_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

$$b_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy \quad \text{——— (4)}$$

thus,

$$\boxed{u = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi x}{b}\right) \cdot \sin\left(\frac{n\pi y}{b}\right)}$$

Ans.

where

$b_n$ is given by eq$^n$ 4