Section-1

Solution :- 3 ⇒

One dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{m} \quad -(1)$$

T = tension in the string.

m = mass per unit length of the string.

Solution of one dimensional wave equation is done by method of separation of variables.

Let $u = X(x)\,T(t)$

where X is a function of x only and T is a function of t only.

Differentiating eqn (2) partially w.r.t x and t respectively and putting the values in eqn. (1)

$$\frac{\partial^2 u}{\partial t^2} = X\frac{\partial^2 T}{\partial t^2} \text{ and}$$

$$\frac{\partial^2 u}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

From one dimensional wave eqn.

$$X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2}$$

Separating the variables,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2 \text{ or } k^2 \text{ or } 0 \text{ (say)}$$

Case (i)

or $\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$ and $\frac{\partial^2 T}{\partial t^2} + k^2 c^2 T = 0$

or $(D^2 + k^2)X = 0$ and $(D^2 + k^2c^2)T = 0$

Auxiliary eq. are $m^2 + k^2 = 0$ and $m^2 + k^2c^2 = 0$

$m = \pm ki$ and $m = \pm kci$

thus complementary function are :-

$X = C_1 \cos kx + C_2 \sin kx$

$T = C_3 \cos kct + C_4 \sin kct$

$u = XT$

⇒ $u = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos kct$

⇒ $+ C_4 \sin kct)$

Case ii :- $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = k^2$

$m = k^2$ and $m^2 = \pm k^2c^2$

$m = \pm k$ and $m = \pm kc$

⇒ $X = C_5 e^{kx} + C_6 e^{-kx}$

and $T = C_7 e^{kct} + C_8 e^{-kct}$

Thus $u = (C_5 e^{kx} + C_6 e^{-kx})(C_7 e^{kct} + C_8 e^{-kct})$

Case iii :- $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = 0$

$m = 0, 0$ and $m = 0, 0$

⇒ $X = C_9 + C_{10}x$ and $T = C_{11} + C_{12}t$

Thus $u = (C_9 + C_{10}x)(C_{11} + C_{12}t)$

thus. the solution given by eqn. satisfies the one dimensional wave equation (2) thus the required solution of one dimensional wave is given by eqn - (3). Now to find the values of C_1, C_2, C_3 and C_4, which are obtained by applying boundary and initial condition