Solution-2) Section-1

Let the eqn. of the temperature distribution is:-

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

$$u = X(x)\,T(t)$$

$$X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T}\frac{\partial T}{\partial t} = -k^2 \text{ (let)}$$

$$X = C_1 \cos kx + C_2 \sin kx$$

$$T = C_3 e^{-k^2 c^2 t}$$

thus $u = (A_n \cos kx + B_n \sin kx)e^{-k^2c^2t}$ --- (2)

Given boundary and initial condition are,

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$$

$$u(x,0) = x(a-x) \quad 0<x<a$$

$$\frac{\partial u}{\partial x} = (-kA_n \sin kx + B_n k \cos kx)e^{-k^2c^2t}$$

$$0 = B_n k e^{-k^2c^2t}$$

$$B_n = 0$$

From eq (2)

$$u = A_n \cos kx\, e^{-k^2c^2t} \quad \text{--- (3)}$$

$$\frac{\partial u}{\partial x} = -kA_n \sin kx\, e^{-k^2c^2t}$$

$$0 = kA_n \sin ka\, e^{-k^2c^2t}$$

$$k = \frac{n\pi}{a}$$

$$u = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right)e^{-n^2\pi^2c^2t/a^2} \quad \text{--- (4)}$$

Now at $t=0$,

$$u(x,0) = \sum A_n \cos\left(\frac{n\pi x}{a}\right)$$

$$A_n = \frac{2}{a}\int_0^a u(x,0)\cos\left(\frac{n\pi x}{a}\right)dx$$

$$A_n = \frac{2}{a}\int_0^a x(a-x)\cos\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\left[(ax-x^2)\left(\frac{a}{\pi x}\sin\frac{n\pi x}{a}\right) - (a-2x)\left(\frac{-a^2}{n^2\pi^2}\cos\frac{n\pi x}{a}\right) + (-2)\left(\frac{-a^3}{n^3\pi^3}\sin\frac{n\pi x}{a}\right)\right]_0^a$$

$$A_n = \frac{2}{a}\left(-(a)\right)\left(\frac{-a^2}{n^2\pi^2}\cos n\pi\right) + a\left(\frac{-a^2}{n^2\pi^2}\right)$$

$$\Rightarrow \frac{-2a^2}{n^2\pi^2}\left[1+\cos n\pi\right]$$

$$\Rightarrow \text{thus } u(x,t) = \sum_0^{\infty} \frac{-2a^2}{n^2\pi^2}(1+\cos n x)\cos\left(\frac{n\pi x}{a}\right)e^{-n^2\pi^2c^2t/a^2}$$