Section-A

Solution-1) Given laplace eqn is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad -(1)$$

Let $u = XY$ where X is a function of x only and Y is a function of y only.

$$\frac{\partial^2 u}{\partial x^2}=Y\frac{\partial^2 X}{\partial x^2}$$

and $$\frac{\partial^2 u}{\partial y^2}=X\frac{\partial^2 Y}{\partial y^2}$$

from eqn. (1)

$$Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}=0$$

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Case i:- $-\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0$ (say)

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=0$$

and $$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0$$

$$X=c_1x+c_2,\ Y=c_3y+c_4$$

At $y=0, Y=0 \Rightarrow c_4=0$

Also $y=b, Y=0 \Rightarrow c_3=0$

$\therefore Y=0$

Thus $u=XY=X(0)$

$u=0$ (not possible)

Case ii:- $-\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=k^2$ (say)

$$\frac{\partial^2 X}{\partial x^2}+k^2X=0$$

and $$\frac{\partial^2 Y}{\partial y^2}-k^2Y=0$$

$$X=c_1\cos kx+c_2\sin kx,\ Y=c_3e^{ky}+c_4e^{-ky}$$

If $y=0, Y=0$

$$c_3+c_4=0$$

and $Y=0$ at $y=b$

$$0=c_3e^{kb}-c_3e^{-kb}$$

$$c_3(e^{kb}-e^{-kb})=0$$

$c_3=0, c_4=0, Y=0$ (Not possible)

Case iii) ⇒ $-\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-k^2$ (say)

$$X=c_1e^{kx}+c_2e^{-kx},$$

$$Y=c_3\cos ky+c_4\sin ky$$

At $y=0, Y=0, c_3=0$

$$Y=c_4\sin ky$$

At $y=b, Y=0$

$$0=c_4\sin kb$$

$$\sin kb=0$$

$$kb=n\pi$$

thus ⇒ $u=(c_1e^{kx}+c_2e^{-kx})c_4\sin\frac{n\pi y}{b}$ (2)

At $x=0, u=0$

$$0=(c_1+c_2)c_4\sin\frac{n\pi y}{b}$$

$$c_1+c_2=0$$

$$c_2=-c_1$$

from eqn (2) $u=\frac{2}{2}c_4c_1(e^{kx}-e^{-kx})\sin\frac{n\pi y}{b}$

$$u=\sum_{n=0}^{\infty} b_n\left(\frac{e^{\frac{n\pi x}{b}}-e^{-\frac{n\pi x}{b}}}{2}\right)\sin\left(\frac{n\pi y}{b}\right) \quad -(3)$$

Let $b_n=2c_1c_4$

At $x=a, u=f(y)$

from eqn (3)

$$f(y)=\sum_{n=0}^{\infty} b_n\left(\frac{e^{\frac{n\pi a}{b}}-e^{-\frac{n\pi a}{b}}}{2}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$f(y)=\sum_{n=0}^{\infty} b_n\sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$b_n\sinh\left(\frac{n\pi a}{b}\right)=\frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy$$

$$b_n=\frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy \quad -(4)$$

Thus $u=\sum_{n=0}^{\infty} b_n\sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$

where b_n is given by eqn (4)