4. $x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 y}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4.$

Put $x = e^X$, $y = e^Y$ so that $X = \log x$ and $Y = \log y$

and let $D = \frac{\partial}{\partial X}$, $D' = \frac{\partial}{\partial Y}$

$$DD' = \frac{\partial^2}{\partial X \partial Y}$$

$$[D(D-1) - 4DD' + 4D'(D'-1) + 6D']z = e^{3X+4Y}$$

$$= [D^2 - 4DD' + 4D'^2] - (D - 2D')]z = e^{3X+4Y}$$

$$= (D - 2D')(D - 2D' - 1)z = e^{3X+4Y}$$

$$CF = f_1(Y + 2X) + e^X f_2(Y + 2X)$$

$$= f_1(\log y + 2\log x) + x f_2(\log y + 2\log x)$$

$$= f_1(\log yx^2) + x f_2(\log yx^2) = \phi_1(yx^2) + x\phi_2(yx^2)$$

$$PI = \frac{1}{D - 2D' - 1}\left[\frac{1}{D - 2D'} e^{3X+4Y}\right]$$

$$= \frac{1}{D - 2D' - 1}\left[\frac{1}{3 - 8}\int e^u du\right] \text{ where } u = 3X + 4Y$$

$$= \frac{1}{D - 2D' - 1}\left[-\frac{1}{5} e^{3X+4Y}\right]$$

$$= -\frac{1}{5}\left[\frac{1}{D - 2D' - 1} e^{3X+4Y}\right]$$

$$= -\frac{1}{5}\left[\frac{1}{3 - 8 - 1} e^{3X+4Y}\right] = \frac{1}{30} e^{3X+4Y} = \frac{1}{30} x^3y^4$$

Hence the complete solution is

$$z = CF + PI = \phi_1(yx)^2 + x\phi_2(yx)^2 + \frac{1}{30}x^3y^4$$