3. One dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{m} \longrightarrow (1)$$

T = Tension in the string

m = Mass per unit length of the string

Let $= X(x) T(t) \longrightarrow (2)$

$$\frac{\partial^2 u}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2} \text{ and}$$

$$\frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$$

From one dimensional wave eq.

$$X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

Saperating the variables,

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = -k^2 \text{ or } k^2 \text{ or } 0 \text{ (say)}$$

Case I :- when $\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k^2$ and $\frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = -k^2$

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0 \text{ and } \frac{\partial^2 T}{\partial t^2} + k^2 c^2 T = 0$$

$$(D^2 + k^2) X = 0 \text{ and } (D^2 + k^2 c^2) T = 0$$

Auxiliary eq. are $m^2 + k^2 = 0$ and $m^2 + k^2 c^2 = 0$

$$m = \pm ki \text{ and } = \pm kci$$

$$X = c_1 \cos kx + c_2 \sin kx$$

$$T = c_3 \cos kct + c_4 \sin kct$$

$$u = XT$$

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos kct + c_4 \sin ckt) \longrightarrow (3)$$

Case ii

When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = k^2$

$m = k^2$ and $m^2 = k^2c^2$

$m = \pm k$ and $m \pm kc$

$X = C_5 e^{kx} + C_6 e^{-kx}$

$T = C_7 e^{kct} + C_8 e^{-kct}$

$u = (C_5 e^{kx} + C_6 e^{-kx})(C_7 e^{kct} + C_8 e^{-kct}) - (4)$

Case iii

When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = 0$

$m = 0, 0$ and $m = 0, 0$

$X = C_9 + C_{10}x$ and $T = C_{11} + C_{12}t$

Thus $u = (C_9 + C_{10}x)(C_{11} + C_{12}t) - (5)$