

Q.2

Ans

Let the eqⁿ of the temperature distribution is:-

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- eqⁿ (1)}$$

$$u = X(x) T(t)$$

$$X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T} \frac{\partial T}{\partial t} = -k^2$$

$$X = C_1 \cos kx + C_2 \sin kx$$

$$T = C_3 e^{-k^2 c^2 t}$$

Thus, $u = (A_n \cos kx + B_n \sin kx) e^{-k^2 c^2 t} \quad \text{--- eqⁿ (2)}$

Given boundary and initial conditions are

$$\left(\frac{\partial u}{\partial x} \right)_{x=0} = 0$$

$$\left(\frac{\partial u}{\partial x} \right)_{x=a} = 0$$

$$u(x, 0) = x(a-x), \quad 0 < x < a$$

$$\frac{\partial u}{\partial x} = (-k A_n \sin kx + B_n k \cos kx) e^{-k^2 c^2 t}$$

$$0 = B_n k e^{-k^2 c^2 t}$$

$$B_n = 0$$

From eqⁿ. (2)

$$u = A_n \cos kx e^{-k^2 c^2 t} \quad \text{--- eqⁿ (3)}$$

$$\frac{\partial u}{\partial x} = k A_n \sin kx e^{-k^2 c^2 t}$$

$$0 = k A_n \sin ka e^{-k^2 c^2 t}$$

$$k = \frac{n\pi}{a}$$

$$u = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right) e^{-n^2 \pi^2 c^2 t/a^2} \quad \text{--- eqn (4)}$$

Now at $t = 0$,

$$u(x, 0) = \sum A_n \cos\left(\frac{n\pi x}{a}\right)$$

$$A_n = \frac{2}{a} \int_0^a u(x, 0) \cos\left(\frac{n\pi x}{a}\right) dx$$

$$A_n = \frac{2}{a} \int_0^a x(a-x) \cos\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a} \left[(ax - x^2) \left(\frac{a}{n\pi} \sin \frac{n\pi x}{a} \right) - (a-2x) \left(\frac{-a^2}{n^2 \pi^2} \cos \frac{n\pi x}{a} \right) + (-2) \left(\frac{-a^3}{n^3 \pi^3} \sin \frac{n\pi x}{a} \right) \right]_0^a$$

$$A_n = \frac{2}{a} \left[-(-a) \left(\frac{-a^2}{n^2 \pi^2} \cos n\pi \right) + a \left(\frac{-a^2}{n^2 \pi^2} \right) \right]$$

$$= \frac{-2a^2}{n^2 \pi^2} [1 + \cos n\pi]$$

Thus,

$$u(x, t) = \sum_0^{\infty} \frac{-2a^2}{n^2 \pi^2} (1 + \cos n\pi) \cos\left(\frac{n\pi x}{a}\right) e^{-n^2 \pi^2 c^2 t/a^2}$$

Ans