Section 5

Question 2
Answer 2. Since R and G are negligible, transmission line equations becomes

$$\frac{\partial e}{\partial x} = -L\frac{\partial i}{\partial t} \qquad (i)$$

and

$$\frac{\partial i}{\partial x} = -C\frac{\partial e}{\partial t} \qquad (ii)$$

For elimination of i, differential eq. (i) partially w.r.t. x and eq (ii) partially w.r.t t we have.

$$\frac{\partial^2 e}{\partial x^2} = -L\frac{\partial^2 t}{\partial x\partial t} \text{ and } \frac{\partial^2 i}{\partial t\,\partial x} = c\frac{\partial^2 e}{\partial t^2}$$

Hence,

$$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial^2 t} \qquad \text{— (iii)}$$

The initial conditions are $(x,0) = i_0$, $e(x,0) = e_0 \sin\frac{\pi x}{l}$ (iv)

Since the ends are suddenly grounded, the boundary conditions (v) are $e(0,t = e(l,t) = 0$

Also $i = i_0$ (Constant) when $t = 0$

$$\frac{\partial i}{\partial x} = 0 \text{ which gives } \frac{\partial e}{\partial t} = 0 \text{ when } t = 0 \qquad (vi)$$

Now let $e = XT$ be a solution of eq. (iii) where X is a function of t only and T is a function of x only

$$\frac{\partial^2 e}{\partial x^2} = XT \text{ and } \frac{\partial^2 e}{\partial t^2} = XT^2$$

∴ From eq (iii) $X''T = LCXT''$

Separating the variables $\frac{X''}{X} = LC\frac{T''}{T} = -p^2$ (say)

This leads to the ordinary diff. equation.

$$\frac{d^2X}{dx^2} + p^2X = 0 \text{ and } \frac{d^2T}{dt^2} + \frac{p^2}{LC}T = 0$$

$$x = C_1 \cos px + C_2 \sin px$$

$$T = C_3 \cos\frac{pt}{\sqrt{LC}} + C_4 \sin\frac{pt}{\sqrt{LC}}$$

$$e = XT = (C_1 \cos px + C_2 \sin px)$$

$$\left[C_3 \cos\frac{pt}{\sqrt{LC}} + C_4 \sin\frac{pt}{\sqrt{LC}}\right) \quad -(vii)$$

Applying the boundary condition (v) in eq (vii), we get

$$C_1 = 0 \text{ and } p = \frac{n\pi}{l}, \text{ n being an integer}$$

$\therefore$ Eq (vii) becomes

$$e = C_2 \sin\frac{n\pi x}{l}\left(C_3 \cos\frac{n\pi t}{l\sqrt{LC}} + C_4 \sin\frac{n\pi t}{l\sqrt{LC}}\right)$$

or

$$e = \sin\frac{n\pi x}{l}\left(A \cos\frac{n\pi t}{l\sqrt{LC}} + B \sin\frac{n\pi t}{l\sqrt{LC}}\right) \quad (viii)$$

where $A = C_2C_3$ and $B = C_2C_4$

$$\frac{\partial e}{\partial t} = \sin\frac{n\pi x}{l}\left(\frac{-An\pi}{l\sqrt{LC}} \sin\frac{n\pi t}{l\sqrt{LC}} + \frac{Bn\pi}{l\sqrt{LC}} \cos\frac{n\pi t}{l\sqrt{LC}}\right)$$

Since $\frac{\partial e}{\partial t} = 0$ where $t = 0$, we get

$$B = 0$$

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∴ From eq (viii)

$$e = A \sin \frac{n\pi x}{l} \cos \frac{n\pi x}{l\sqrt{LC}}$$

By Superposition, $e = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi x}{l\sqrt{LC}}$ is also a solution

But $e = e_0 \sin \frac{n x}{l}$ when $t = 0$

∴

$$e_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

$$A_1 = e_0 \text{ and } A_2 = A_3 = \ldots\ldots = 0$$

Hence, $e = e_0 . \sin \frac{n x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$

Now, $-L \frac{\partial i}{\partial t} = \frac{\partial e}{\partial x}$

$$\frac{\partial i}{\partial t} = -\frac{1}{L}, \frac{e_0 \pi}{l} \cos \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$$

Integrating w.r.t. t, regarding x as constant

$$i = \frac{e_0 \pi}{Ll} \cos \frac{\pi x}{l} . \frac{l\sqrt{LC}}{\pi} \sin \frac{\pi t}{l\sqrt{LC}} + f(x) \quad (ix)$$

where f(x) is an arbitary constant function.

Since $i = i_0$, when $t = 0$, we have $i_0 = 0 + f(x)$ or $f(x) = i_0$

∴ From eq (ix) we have

$$i = i_0 - e_0 \sqrt{\frac{C}{L}} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{LC}}$$