Question 4. Solve the linear differential equation,

$$x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} = x^2 y^2$$

Answer

Put $x = e^X, y = e^Y$ so that $X = \log x$ and $Y = \log y$ and let $D = \frac{\partial}{\partial X}, D' = \frac{\partial}{\partial Y}$

and $DD' = \frac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to.

$$[D(D-1) - 4DD' + 4D'(D'-1) + 6D']z = e^{8x - 6Y}$$

$$= [(D^2 - 4DD' + 4D^2) - (D - 2D')]z = e^{8x - 4Y}$$

$$= (D - 2D')(D - 2D' - 1)z = e^{8X + 4Y}$$

Its $\quad CF = f_1(Y + 2X) + e^{x/2}(Y + 2X)$

$$= f_1(\log Y + 2\log x) + x f_2(\log y + 2\log x)$$

$$= f_1(\log yx^2) + x f_2(\log yx^2) = g_1(yx^2)) + x g_2(yx)^2$$

$$PI = \frac{1}{D - 2D' - 1}\left[\frac{1}{D - 2D} e^{2X - 4x}\right]$$

$$= \frac{1}{D - 2D' - 1}\left[\frac{1}{3 - 8}\int e^x du\right] \text{ where } 3x + 4X = 4$$

$$= \frac{1}{D - 2D' - 1}\left[-\frac{1}{5} e^{3X - 4Y}\right]$$

$$= -\frac{1}{5}\left[\frac{1}{D - 2D' - 1} e^{3X + 4X}\right]$$

$$= -\frac{1}{5}\left[\frac{1}{3 - 8 - 1} e^{3x + 4x}\right] = \frac{1}{30} e^{3X + 4X}$$

$$= \frac{1}{30} x^2 y^3$$

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Hence the Complete Solution is

$$z = CF + PI = g_1(yx^2) + xg_3(yx^3) + \frac{1}{30}x^2y$$

where g_1 and g_2 are arbitrary function.