Section 1

Question 3. Write the solution of one dimensional wave equation.

Answer One dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{m}$$

T = Tension in string

m = Mass per unit length of the string.

Solution of one dimentional wave equation is done by method of sepration of variables.

Let
$$u = X(x)\,T(t)$$

where X is a function of x and T is a function of t only.

Differentiating eq. (2.9.2) partially w.r.t. x and t respectively and putting the value in eq. (2.9.1)

$$\frac{\partial^2 u}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2} \quad \text{and}$$

$$\frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$$

From one dimensional wave equation,

$$X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

Separating the variables,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2 \text{ or } k^2 \text{ or } 0 \text{ (say)}$$

Case i: when $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2$

or $\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$ and $\frac{\partial^2 T}{\partial t^2} + k^2 c^2 T = 0$

or $(D^2 + k^2)X = 0$ and $(D^2 + k^2 c^2)T = 0$

Auxiliary equations are $m^2 + k^2 = 0$ and $m^2 + k^2 c^2 = 0$

$m = \pm ki$ and $m = \pm kCi$

Thus Complementary functions are

$X = C_1 \cos kx + C_2 \sin kx$

$T = C_3 \cos kCt + C_4 \sin kCt$

$u = XT$

$u = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos kCt + C_4 \sin kCt)$

Case ii: when $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = k^2$

$m^2 = k^2$ and $m^2 = k^2 c^2$

$m = \pm k$ and $m = \pm kC$

$X = C_5 e^{kx} + C_6 e^{-kx}$

and $T = C_7 e^{kCt} + C_8 e^{-kCt}$

$u = (C_5 e^{kx} + C_6 e^{-kx})(C_7 e^{kCt} + C_8 e^{-kCt})$

Case (iii) when $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$ and $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = 0$

$$m = 0, 0 \text{ and } m = 0, 0$$

$$X = C_9 + C_{10} x \text{ and } T = C_{11} + C_{12} t$$

Thus $$u = (C_9 + C_{10} x)(C_{11} + C_{12} t)$$

The solution given by (29.3) satisfies the one dimensional wave equation. Thus the required solution of one dimensional wave is given by eq (29.3) Now to find the value of C_1, C_2, C_3 and C_4 which are obtain by applying boundary and initial conditions.