Section - 1

Q1 Solve the laplace equation

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle in the xy plane, $0 \le y \le b$ satisfying the boundary condition $u(x,0)=0$, $u(x,b)=0$ and $u(a,y)=f(y)$

Ans Given laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

Let $u = XY$, where X is a function of x only Y is a function of y only.

$$\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = X \frac{\partial^2 Y}{\partial y^2}$$

from eq (1)

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$-\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}$$

Case (i): $-\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = 0$ (say)

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = 0$$

and $\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = 0$

$$X = C_1 x + C_2, \quad Y = C_3 y + C_4$$

At $y=0,\ Y=0 \Rightarrow C_4 = 0$
$y=b,\ Y=0 \Rightarrow C_3 = 0$
$Y = 0$

Thus, $U = XY = X(0)$
$U = 0$ (not possible)

Case (ii): $-\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = K^2$ (say)

$$\frac{\partial^2 X}{\partial x^2} + K^2 X = 0$$

and $\frac{\partial^2 Y}{\partial y^2} - K^2 Y = 0$

If $X = C_1 \cos Kx + C_2 \sin Kx$, $Y = C_3 e^{Ky} + C_4 e$
$y=0,\ Y=0$

$C_3 + C_4 = 0$

$C_4 = -C_3$

$y = 0$ at $y = b$

$0 = C_3 e^{kb} - C_3 e^{-kb}$

$C_3(e^{kb} - e^{-kb}) = 0$

$C_3 = 0 \quad C_4 = 0, \; Y = 0$ (Not possible)

Case (iii) $-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2$ (say).

$X = C_1 e^{kx} + C_2 e^{-kx}$,

$Y = C_3 \cos ky + C_4 \sin ky$

$y = 0, \; Y = 0, \; C_3 = 0$

$Y = C_4 \sin ky$

$y = b \quad Y = 0$

$0 = C_4 \sin kb$

$\sin kb = 0$

$kb = n\pi$

$k = \frac{n\pi}{b}$

Thus $\quad u = (C_1 e^{kx} + C_2 e^{-kx}) C_4 \sin \frac{n\pi y}{b} \quad (2)$

$x = 0, \; u = 0$

$0 = (C_1 + C_2) C_4 \sin\left(\frac{n\pi y}{b}\right)$

$C_1 + C_2 = 0$

$C_2 = -C_1$

from eq (2)

$$u = \frac{2}{2} C_4 C_1 (e^{kx} - e^{-kx}) \sin \frac{n\pi y}{b}$$

$$u = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}}}{2}\right) \sin\left(\frac{n\pi y}{b}\right)$$

from eq (3)

$$f(y) = \sum_{n=0}^{\infty} b_n \left|\frac{e^{\frac{n\pi a}{a}} - e^{\frac{-n\pi a}{b}}}{2}\right| \sin\left(\frac{n\pi y}{b}\right)$$

$$f(y) = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$b_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy \quad (4)$$

$$u = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

where b_n is given by eq (4). Ans