$\frac{\partial y}{\partial b}$ — (2)

(Q2) Find the temperature distribution in a rod of length 'a' which is perfectly insulated including the ends and the initial temperature distribution is $x(a-x)$, $0<x<a$.

Sol$^n$

let the equation of the temperature distribution is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{— (1)}$$

— (3)

$$u = X(x)T(t)$$

$$X\frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

Page :
Date: / /

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T}\frac{\partial T}{\partial t} = -k^2 \text{ (let)}$$

$$X = c_1 \cos kx + c_2 \sin kx$$
$$T = c_3 e^{-k^2c^2t}$$

Thus, $u = (A_n \cos kx + B_n \sin kx) e^{-k^2c^2t}$ — (2)

Given boundary and initial conditions are

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0$$

$$u(x,0) = x(a-x), \quad 0 < x < a$$

$$\frac{\partial u}{\partial x} = (-kA_n \sin kx + B_n k \cos kx) e^{-k^2c^2t}$$

$$0 = B_n \cancel{k} e^{-k^2c^2t}$$

$$B_n = 0$$

from eq (2)

$$u = A_n \cos kx \, e^{-k^2c^2t} \quad \text{— (3)}$$

$$\frac{\partial u}{\partial x} = kA_n \sin kx \, e^{-k^2c^2t}$$

$$0 = kA_n \sin ka \, e^{-k^2c^2t}$$

$$k = \frac{n\pi}{a}$$

$$u = \sum_{n=0} A_n \cos\left(\frac{n\pi x}{a}\right) e^{-n^2\pi^2c^2t/a^2} \quad \text{— (4)}$$

Now at $t=0$

$$u(x,0) = \sum A_n \cos\left(\frac{n\pi x}{a}\right)$$

$$A_n = \frac{2}{a}\int_0^a u(x,0) \cos\left(\frac{n\pi x}{a}\right) dx$$

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Page :

Date: / /

$$A_n = \frac{2}{a}\int_0^a x(a-x)\cos\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\Big[(ax-x^2)\left(\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right)-(a-2x)\left(\frac{-a^2}{n^2\pi^2}\cos\frac{n\pi x}{a}\right)+(-2)\left(\frac{-a^3}{n^3\pi^3}\sin\frac{n\pi x}{a}\right)\Big]_0^a$$

$$A_n = \frac{2}{a}\left[-(-a)\left(\frac{-a^2}{n^2\pi^2}\cos n\pi\right)+a\left(\frac{-a^2}{n^2\pi^2}\right)\right]$$

$$= \frac{2a^2}{n^2\pi^2}\left[1+\cos n\pi\right]$$

thus,

$$u(x,t) = \sum_0^{\infty} -\frac{2a^2}{n^2\pi^2}(1+\cos n\pi)\cos\left(\frac{n\pi x}{a}\right)e^{-n^2\pi^2c^2t/a^2}$$

(2)

e

$)e^{-k^2c^2t}$