SEC - 1

Given Laplace equation is

(1) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{———— (1)}$$

Let $u = XY$, where X is a function of x only Y is a function of y only.

$$\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2}$$

and $$\frac{\partial^2 u}{\partial y^2} = X \frac{\partial^2 Y}{\partial y^2}$$

from eq (1)

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Case (i) $$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \text{ (say)}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$$

and $$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$X = c_1 x + c_2, \; Y = c_3 y + c_4$$

At

$y = 0, Y = 0 \Rightarrow c_4 = 0$

Also $y = b, Y = 0 \Rightarrow c_3 = 0$

$\therefore Y = 0$

Thus $u = XY = X(0)$

$u = 0$ (not possible)

Case (ii) $$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 \text{ (say)}$$

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$$

and $$\frac{\partial^2 Y}{\partial y^2} - k^2 Y = 0$$

$X = c_1 \cos kx + c_2 \sin kx$

$Y = c_3 e^{ky} + c_4 e^{-ky}$

$y = 0, Y = 0$

if $c_3 + c_4 = 0$

$c_4 = -c_3$

and $Y = 0$ at $y = b$

$0 = c_3 e^{kb} - c_3 e^{-kb}$

$c_3(e^{kb} - e^{-kb}) = 0$

$c_3 = 0, c_4 = 0 \quad Y = 0$ (Not possible)

Case (iii) $-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$ (Say)

$X = c_1 e^{kx} + c_2 e^{-kx}$

$Y = c_3 \cos ky + c_4 \sin ky$

At $y = 0,\ Y = 0,\ c_3 = 0$

$Y = c_4 \sin ky$

$y = b,\ Y = 0$

$0 = c_4 \sin kb$

$\sin kb = 0$

$kb = n\pi$

$k = \frac{n\pi}{b}$

Thus $u = (c_1 e^{kx} + c_2 e^{-kx}) c_4 \sin\frac{n\pi y}{b}$ — (II)

At $x = 0,\ u = 0$

$0 = (c_1 + c_2) c_4 \sin\frac{n\pi y}{b}$

$c_1 + c_2 = 0$

$c_2 = -c_1$

from eq (II)

$u = \frac{2}{2} c_4 c_1 (e^{kx} - e^{-kx}) \sin\frac{n\pi y}{b}$

$u = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}}{2}\right) \sin\left(\frac{n\pi y}{b}\right)$ — (III)

Let $b_n = 2c_1 c_4$

At $x = a,\ u = f(y)$

from eq (3) $f(y) = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}}{2}\right) \sin\left(\frac{n\pi y}{b}\right)$

$f(y) = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$

$b_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$

$b_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$ — (IV)

Thus, $u = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$

where b_n is given by eq (IV)