(Q3) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to boundary conditions $u(0,y) = 0 = u(\pi, y)$, & $u(x,0) = u_0$, $\lim_{y\to\infty} u(x,y) = 0$, $0 < x < \pi$.

Sol^n

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

let $u = XY$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

u.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{-Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (let)}$$

$X = C_1 \cos kx + C_2 \sin kx$, $Y = C_3 e^{ky} + C_4 e^{-ky}$

$$u = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad \text{---} (1)$$

$u(0,y) = 0$

$0 = C_1 (C_3 e^{ky} + C_4 e^{-ky})$

$C_1 = 0$

Page :
Date: / /

from eq^n ①

$$u = \sin kx (A_n e^{ky} + B_n e^{-ky}) \quad \text{②}$$

$$u(\pi, y) = 0$$

$$\sin k\pi = 0 \Rightarrow k = n$$

$$u = \sin nx (A_n e^{ny} + B_n e^{-ny}) \quad \text{③}$$

$\lim_{y \to \infty} u(x,y) = 0$, it satisfies only when $A_n = 0$

from eq^n ③

$$u = \sum B_n e^{-ny} \sin nx$$

now;

$$u(x, 0) = u_0$$

$$u_0 = \sum B_n \sin nx$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} u_0 \sin nx \, dx = \frac{-2u_0}{\pi}$$

$$\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-2u_0}{\pi} \left\{\frac{(-1)^n - 1}{n}\right\}$$

Thus from eq^n ④

$$u = \sum \frac{-2u_0}{\pi n} [(-1)^n - 1] e^{-ny} \sin nx$$