(Q3) Write the solution of one dimensional wave equation.

Sol^n

One dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{m} \quad \text{--- (1)}$$

T = tension in the string.

m = Mass per unit length of the string.

Sol^n of one dimensional wave eq^n is done by method of separation of variables.

Let $u = X(x)\, T(t)$ --- (2)

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Diff eq ② partially w.r.t x & t respectively & putting the value in eq ①

$$\frac{\partial^2 u}{\partial t^2} = X\frac{\partial^2 T}{\partial t} \;\&$$

$$\frac{\partial^2 u}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

from one dimensional wave eqn.

$$X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2}$$

Seperating the variables

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2 \text{ or } k^2 \text{ or } 0 \text{ (say)}$$

Case 1: When $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2$ & $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2$

or $\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$ & $\frac{\partial^2 T}{\partial t^2} + k^2 c^2 T = 0$

$$(D^2 + k^2)(X) = 0 \;\&\; (D^2 + k^2 c^2)T = 0$$

Auxiliary eqn are: $m^2 + k^2 = 0$ & $m^2 + k^2c^2 = 0$

$$m = \pm ki \;\&\; m = \pm kci$$

Thus complementary functions are

$$X = c_1 \cos kx + c_2 \sin kx$$

$$T = c_3 \cos kct + c_4 \sin kct$$

$$u = XT$$

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos kct + c_4 \sin$$

Case ii:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2 \;\&\; \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = k^2$$

$$m = k^2 \;\&\; m^2 = k^2 c^2$$

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...w.r.t x & t

...e value in eq ②

...ave eqn.

$\frac{\partial^2 T}{\partial t^2} = -k^2$ or k^2 or 0 (say)

$\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2$

$+ k^2 c^2 T = 0$

$k^2 c^2)T = 0$

$^2 = 0$ & $m^2 + k^2 c^2 = 0$

$m = \pm kci$

...ctions are

...n kx

sin kct

$\ldots n kx)(c_3 \cos kct + c_4 \sin kct)$ — ③

$\frac{\partial^2 T}{\partial t^2} = k^2$

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$m = \pm k$ & $m = \pm kc$

$X = c_5 e^{kx} + c_6 e^{-kx}$

$T = c_7 e^{kct} + c_8 e^{-kct}$

$u = (c_5 e^{kx} + c_6 e^{-kx})(c_7 e^{kct} + c_8 e^{-kct})$ — ④

case 3: when $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$ & $\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = 0$

$m = 0, 0$ & $m = 0, 0$

$X = c_9 + c_{10} x$ & $T = c_{11} + c_{12} t$

$u = (c_9 + c_{10} x)(c_{11} + c_{12} t)$ — ⑤

Initially if the string is a rest $g(x) = 0$.
Now to find the constants of eq. ③
apply the boundary condition (i) to eq ③

$0 = c_1 (c_3 \cos kct + c_4 \sin kcT)$

$c_1 = 0$ $[\because c_3 \cos kct + c_4 \sin ct \neq 0]$

from eq. ③

$u = c_2 \sin kx (c_3 \cos kct + c_4 \sin kct)$ — ⑥

now put boundary condition (ii), in eqn ⑥

$0 = c_2 \sin kl (c_3 \cos kct + c_4 \sin kct)$

$\sin kl = 0$

$kl = n\pi$

$k = \frac{n\pi}{l}$

from eq ⑤

$u = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right)$

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$$u = \sin\frac{n\pi x}{l}\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right) \quad -(7)$$

now apply condition (iii) & (iv)

$$u = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = f(x) \quad -(8)$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{t=0}^{\infty}\frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l} = g(x) \quad -(9)$$

The left hand side of the eqn ~~(2.1.8)~~ (8) & eq ~~(2.10)~~ (9) represents fourier sine explonsion of righ hand side. Thus,

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$\frac{n\pi c}{l}B_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

Putting the value of A_n & B_n in eqn (7), we obtain the required soln of one dimensional wave eqn.

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