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(Q1) Solve the laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle in the x-y plane, $0 \le x \le a$ & $0 \le y \le b$ satisfying the following boundary conditions $u(x,0) = 0$, $u(x,b) = 0$ & $u(0,y) = 0$, $u(a,y) = f(y)$.

Soln

Given eqn

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{---} (1)$$

let $u = XY$, where X is a function of x only & Y is a function of y only

$$\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = X \frac{\partial^2 Y}{\partial y^2}$$

from eq (1)

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$-\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}$$

Case (i)

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \text{ (say)}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = 0$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$X = c_1 x + c_2, \quad Y = c_3 y + c_4$$

At, $y = 0,\ Y = 0 \Rightarrow c_4 = 0$

$y = b,\ Y = 0 \Rightarrow c_3 = 0$

$Y = 0$

$u = XY = X(0)$

$u = 0$ (not possible.

Case(iii): $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2$ (say)

$\cancel{X = c_1 e^{kx} +}$

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} - k^2 Y = 0$$

$X = c_1 \cos kx + c_2 \sin kx,$

$Y = c_3 e^{ky} + c_4 e^{-ky}$

$y = 0,\ Y = 0$

$c_3 + c_4 = 0$

$c_4 = -c_3$

$Y = 0$ at $y = b$

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$$0 = C_3 e^{kb} - C_3 e^{-kb}$$

$$C_3(e^{kb} - e^{-kb}) = 0$$

$$C_3 = 0,\ C_4 = 0,\ Y = 0$$

Case iii,

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \text{ (say)}$$

$$X = C_1 e^{kx} + C_2 e^{-kx}$$

$$Y = C_3 \cos ky + C_4 \sin ky$$

$$y = 0,\ Y = 0,\ C_3 = 0$$

$$Y = C_4 \sin ky$$

$$y = b,\ Y = 0$$

$$0 = C_4 \sin kb$$

$$\sin kb = 0$$

$$kb = n\pi$$

$$k = \frac{n\pi}{b}$$

$$u = (C_1 e^{kx} + C_2 e^{-kx})\, C_4 \sin\frac{n\pi y}{b} \quad \text{— (2)}$$

$$x = 0,\ u = 0$$

$$0 = (C_1 + C_2)\, C_4 \sin\frac{n\pi y}{b}$$

$$C_1 + C_2 = 0$$

$$C_2 = -C_1$$

from eq (2)

$$u = \frac{2}{2} C_4 C_1 (e^{kx} - e^{-kx}) \sin\frac{n\pi y}{b}$$

$$u = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}}{2}\right) \sin\left(\frac{n\pi y}{b}\right) \quad \text{— (3)}$$

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$b_n = 2c_1c_4$

$x = a,\ u = f(y)$

from eq^n 3

$$f(y) = \sum_{n=0}^{\infty} b_n \left(\frac{e^{\frac{n\pi a}{b}} - e^{\frac{-n\pi a}{b}}}{2} \right) \sin\left(\frac{n\pi y}{b}\right)$$

$$f(y) = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$b_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

$$b_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy \quad \text{— (4)}$$

Thus:

$$u = \sum_{n=0}^{\infty} b_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

Where b_n is given by eq 4

x