Q State and prove the cauchy integral Test?
maclaurin's

Ans Cauchy-Integral Test:

Improper integrals: Integrals of the form $\int_a^{\infty} f(x)\,dx$

where $a \in R$ are called improper integrals

Let $F(t) = \int_a^{t} f(x)\,dx$ for $a \le t < \infty$

If $\lim_{t \to \infty} F(t)$ exists and is equal to $l \in R$, the improper integral $\int_a^{\infty} f(x)\,dx$ is said to converage to l, otherwise it is called a divergent integral.

Theorm: Let $f(x)$ be a non-negative monotonically decreasing integrable function on $[1, \infty]$.

Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\,dx$ converges or diverge together i.e, the series $\sum f(n)$ converges or diverges according as the integral $\int_1^{\infty} f(x)\,dx$ tends to a fine limit or diverges to ∞ as $n \to \infty$.

Proof: Since $f(x)$ is non-negative on $[1, \infty]$

therefore $f(x) \geq 0 \ \forall x \geq 1$

i.e the series $\sum_{n=1}^{\infty} f(n)$ is non-negative terms

For any $x \in [1, \infty[$, we can find $n \in \mathbb{N}$ such that

$n \leq x \leq n+1$

Since f is monotonically decreasing on $[1, \infty[$

therefore, we have

$$f(n) \geq f(x) \geq f(n+1) \text{ if } n \leq x \leq n+1$$

$$\therefore \int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

or $f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)$