Section - 1

Ans-3 Cauchy's Integral Test:-

Theorem: Let $f(x)$ be a non negative monotonically decreasing integrable function on $[1, \infty]$. Then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\, dx$ converges or diverge together i.e, the series $\sum f(n)$ converges or diverges according as the integral $\int_1^n f(x)\, dx$ tends to a finite limit or diverges to ∞ as $n \to \infty$

Proof: $\int_1^n f(x)\, dx$ tends to a finite limit or diverges to ∞ as $n \to \infty$

Since $f(x)$ is non negative on $[1, \infty[$

Therefore $f(n) \geq 0 \;\forall\; n \geq 1$

i.e. the series $\sum_{n=1}^{\infty} f(n)$ is of non-negative terms.

For any $x \in [1, \infty[$, we can find $n \in N$ such that $n \leq x \leq n+1$.

$f(n) \geq f(x) \geq f(n+1)$ if $n \leq x \leq n+1$

$\int_n^{n+1} f(n)\, dx \geq \int_n^{n+1} f(x)\, dx \geq \int_n^{n+1} f(n+1) dx$

or $f(n) \geq \int_n^{n+1} f(x)\, dx \geq f(n+1) \ldots\ldots (1)$

Putting $n = 1, 2, \ldots (n-1)$ in (1) in succession and then adding all the results, we get

$$f(1) + f(2) + \ldots + f(n-1) \geq \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \ldots + \int_{n-1}^{n} f(x)\,dx \geq f(2) + f(3) + \ldots + f(n)$$

Let $S_n = f(1) + f(2) + \ldots + f(n)$ and ---- (2)

$$I = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \ldots + \int_{n-1}^{n} f(x)\,dx$$

$$= \int_1^n f(x)\,dx.$$

then (2) = $S_n - f(n) \geq I_n \geq S_n - f(1)$

$$-f(n) \geq I_n - S_n \geq -f(1)$$

or $f(n) \leq S_n - I_n \leq f(1)$

result (3) is true for all $n \in N$.

Now $u_{n+1} - u_n = (S_{n+1} - I_{n+1}) - (S_n - I_n)$

$$= (S_{n+1} - S_n) - (I_{n+1} - I_n)$$

$$= f(n+1) - \int_n^{n+1} f(x)\,dx \leq 0 \text{ using (1)}$$

$u_{n+1} \leq u_n$ for all $n \in N$

i.e. $\langle u_n \rangle$ is monotonically decreasing sequence.

Also by (3), $u_n \geq f(n) \geq 0$ for all n and hence $\langle u_n \rangle$ is bounded below. Thus the sequence $\langle u_n \rangle$ is convergent i.e. $\langle u_n \rangle$ tends to infinite limit as $n \to \infty$

Since $S_n = u_n + I_n$ and $\langle u_n \rangle$ is convergent, it follows that the sequences $\langle S_n \rangle$ and $\langle I_n \rangle$ converge or diverge together.

Consequently the series $\Sigma f(n)$ and the integral $\int_1^\infty f(x)\,dx$ converge or diverge together.