Section-1

Ans-2 Taylor's series for $f(z)$ about $z=a$

Statement: If $f(z)$ is analytic inside a circle c with centre at a, for all z inside c.

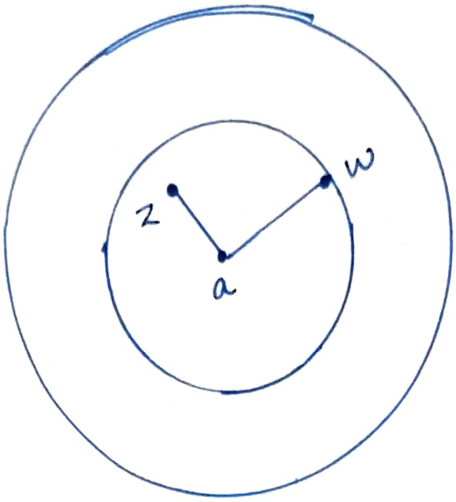
$$f(z) = f(a) + (z-a)\cdot f'(a) + \frac{(z-a)^2}{2!} f''(a)$$

$$+ \ldots + \frac{(z-a)^n}{n!}\cdot f^n(a) + \ldots$$

or $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, where

$$a_n = \frac{f^n(a)}{n!}$$

Proof: Let z be any point inside a circle c. Draw a circle c_1 with centre at a and radius smaller than that of c such that z is an interior point of c_1.



Let w be any point on C_1, then

$|z-a| < |w-a|$ i.e. $\left|\frac{z-a}{w-a}\right| < 1$

Now, $\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[1-\frac{z-a}{w-a}\right]^{-1}$

expanding R.H.S. by binomial theorem as $\left|\frac{z-a}{w-a}\right| < 1$,

we get $\frac{1}{w-z} = \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\dots\dots+\left(\frac{z-a}{w-a}\right)^n + \dots\right]$

This series converges uniformly since $\left|\frac{z-a}{w-a}\right| < 1$.

Multiplying both sides of equation (1) by $\frac{1}{2\pi i} f(w)$

and integrating term by term w.r.t. w, over C_1,

we get

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f w}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint\frac{f w}{(w-a)^2}\,dw$$

$$+\frac{(z-a)^2}{2\pi i}\cdot\oint_{C_1}\frac{f(w)}{(w-a)^3}\cdot dw + \dots +$$

$$\frac{(z-a)^n}{2\pi i} \oint_{C_1} \frac{f(w)\,dw}{(w-a)^{n+1}} + \dots\dots$$

$$\Rightarrow f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

$$+ \frac{(z-a)^n}{n!} f^n(a) + \dots \qquad (1)$$

which is the required Taylor's series for $f(z)$ about $z=a$

If we put $z = a+h$ in eq^n (1) we get,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots$$

If we put $a=0$

the eq^n (1) becomes

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \dots + \frac{z^n}{n!} f^n(0) + \dots$$