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A3 Cauchy - Maclaurin's Integral test →

- Improper integrals : Integrals of the form

$\int_a^{\infty} f(x)\,dx$ where $a \in R$ are called improper integrals

let $f(t) = \int_a^t f(x)\,dx$ for $a \leq t < \infty$

if $\lim_{t \to \infty} f(t)$ exists & is equal to $l \in R$ the improper integral

$\int_a^{\infty} f(x)\,dx$ is said to be convergece to $l$, otherwise it is called a divergent integral

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theorm : Let $f(x)$ be a non-negative monotonically decreasing integrable function on $[1, \infty[$. Then the Series $\sum_{n=1}^{\infty} f(n)$ & the improper integral $\int_1^{\infty} f(x)\,dx$ converges or diverges together i.e., the series $\sum f(n)$ converges & diverges a/c as the integral $\int_1^{y} f(x)\,dx$ tends to a finite limit or diverges to $\infty$ as $y \to \infty$

Proof :- Since $f(x)$ is non-negative on $[1, \infty[$

Therefore $f(x) \geq 0 \;\forall\; x \geq 1$.

i.e., the series $\sum_{n=1}^{\infty} f(n)$ is of non-negative terms.

For any $x \in [1, \infty[$, we can find $n \in \mathbb{N}$ such that $n \leq x < n+1$

Since $f$ is monotonically decreasing on $[1, \infty[$, therefore we have,

$$f(n) \geq f(x) \geq f(n+1) \text{ if } n \leq x \leq n+1.$$

$$\therefore \int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

or,

$$f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)$$

Putting $n = 1, 2, \ldots (n-1)$ in ① & then adding all the results,

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$$f(1)+f(2)+\cdots f(n-1) \geq \int_1^2 f(x)dx + \int_2^3 f(x)dx + \cdots + \int_{n-1}^{n} f(x)dx$$

$$\geq f(2)+f(3) \cdots + f(n) \quad -(2)$$

let, $S_n = f(1)+f(2)+\cdots f(n)$ &

$$I = \int_1^2 f(x)dx + \int_2^3 f(x)dx + \cdots$$

$$\int_{n-1}^{n} f(x)dx = \int_1^n f(x)dx$$

Then (2) can be written as,

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$S_n - f(n) \geq I_n \geq S_n - f(1)$

$-f(n) \geq I_n - S_n \geq -f(1)$.

$f(n) \leq S_n - I_n \leq f(1)$ . — (3)

The result (3) is true for all $n \in \mathbb{N}$

let $u_n = S_n - I_n$ for all $n \in \mathbb{N}$

Now, $u_{n+1} - u_n = (S_{n+1} - I_{n+1}) -$

$(S_n - I_n) = (S_{n+1} - S_n) - (I_{n+1} - I_n)$

$\Rightarrow f(n+1) - \int_n^{n+1} f(x)\,dx \leq 0$, using - (1)