(2) Tayloor's theorem → In calculus,
taylor's theorm
gives an approximation of a k-times
differentiable function around a
given point by a polynomial of degree
k, called the k^{th} order
taylor polynomial. For a smooth
function, the taylor polynomial is a
truncation at the order k of the
taylor series of the function

Date
Page

~~Prove~~ $f(z) = f(a) + (z-a) \cdot f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots + \frac{(z-a)^n}{n!} \cdot f^n(a)$

as $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, where

$a_n = \frac{f^n(a)}{n!}$

Proof: let z be an any point inside the circle C_1 with centre at a and radius smaller than that of C such that z is a interior point of C_1

Date ______

Page ______

Let w be any point on C_1 then,

$$|z-a| < |w-a| \text{ i.e. } \left|\frac{z-a}{w-a}\right| < 1$$

Now,

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[1-\frac{z-a}{w-a}\right]^{-1}$$

Expand RHS By binomial th^m we get,

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\dots+\left(\frac{z-a}{w-a}\right)^n+\dots\right]$$

This series converges uniformly since

$\left|\frac{z-a}{w-a}\right| < 1$, Multiplying both sides of equation (1) by $\frac{1}{2\pi i} f(w)$

and integrating term by term w.r.t. w over C_1,

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-a}\, dw +$$

$$\frac{z-a}{2\pi i}\int \frac{f(w)}{(w-a)^2}\, dw +$$

$$\text{--}\ \frac{(z-a)^n}{2\pi i}\int \frac{f(w)\, dw}{(w-a)^{n+1}} \text{---}$$

$$\Rightarrow \quad f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a)$$

$$+ \dots + \frac{(z-a)^n}{n!} f^n(a) + \dots$$

Which is the required Taylor series for $f(z)$ about $z = a$.