Solution

KAS103

Section-A

1. (a) $\rho(A) = 2$

(b) Let $A = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$

$$A^{*} = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 2 \end{bmatrix}$$

$$\Rightarrow \quad A = A^{*}$$

(c) $z = \sin(y/x)$ homogenious of order '0'

hence ~~$xz_x +$~~ $xz_x + yz_y = 0$

(d) Since $|x|$ is not differentiable in $[-1,1]$
So Rolle's Th. is not applicable.

(e) $$y_n = \frac{2^n \, n!}{(1-2x)^{n+1}}$$

Section-B

2 (a) $[A|B] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

~~On~~ ~~si~~ using elementary transformations & simplification.

$$[A|B] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A|B) = \rho(A) = 3 =$ No. of variables

hence system is consistent & have unique soln.

Equivalent system of eqn. is

$$x + 2y - z = 3$$
$$y = 4$$
$$5z = 20$$

$\Rightarrow \boxed{x = -1,\ y = 4,\ z = 4}$

Ans

2(b)

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 3 & 1-\lambda & 1 \\ 2 & 3 & 2-\lambda \end{vmatrix} = \cancel{\lambda^3 + 4\lambda^2 - 5\lambda - 9} = 0$$

$$= -\lambda^3 + 4\lambda^2 + 5\lambda + 9 = 0$$

$$= \lambda^3 - 4\lambda^2 - 5\lambda - 9 = 0$$

$$\Rightarrow |A-\lambda I| = \lambda^3 - 4\lambda^2 - 5\lambda - 9 = 0$$

by Cayley-Hamilton Th.

$$A^3 - 4A^2 - 5A - 9I = 0$$

$$\begin{bmatrix} 46 & 37 & 38 \\ 47 & 42 & 41 \\ 70 & 69 & 63 \end{bmatrix} - 4\begin{bmatrix} 8 & 8 & 7 \\ 8 & 7 & 9 \\ 15 & 11 & 11 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$-9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

C-H Th. verified.

2 (c) $y = e^{a\sin^{-1}x}$ —— (1) $(y)_0 = 1$ (4)

$y_1 = e^{a\sin^{-1}x}\left(\frac{a}{\sqrt{1-x^2}}\right) \Rightarrow y_1\sqrt{1-x^2} = ay \quad \therefore (y_1)_0 = a(y)_0 = a$

$\Rightarrow (1-x^2)y_1^2 = a^2y^2$

$\therefore 2(1-x^2)y_1y_2 - 2xy_1^2 = 2a^2yy_1$

$(1-x^2)y_2 - xy_1 = a^2y$

Applying Leibnitz Th. to find nth derivative of LHS.

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

$\therefore (y_{n+2})_0 = (n^2+a^2)(y_n)_0$

$y_n(0) = \begin{cases} a^2(2^2+a^2)(4^2+a^2)\cdots[(n-2)^2+a^2] & \text{when } n \text{ is even} \\ a^2(1^2+a^2)(3^2+a^2)\cdots[(n-2)^2+a^2] & \text{when } n \text{ is odd} \end{cases}$

2 (d) CMVT $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} \Rightarrow -e^{a+b} = -e^{2c}$

$\Rightarrow \boxed{a+b=2c} \Rightarrow \boxed{c = \frac{a+b}{2}}$

3(a) $$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

$$\Rightarrow \lambda = 2, 2, -2$$

When $\lambda = -2$

$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0$$
$$x_1 + 3x_2 + x_3 = 0$$

$$X_1 = \begin{bmatrix} 4 \\ +1 \\ -7 \end{bmatrix} K.$$

When $\lambda = 2$:

$$\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \cancel{x_3}\, t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$ as~~at~~

$$\boxed{\lambda = -2, 2, 2, \quad X_1 = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$ Ans

3 b

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \longleftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -7 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$R_1 \longrightarrow R_1 - 2R_2$$
$$R_3 \rightarrow R_3 + 5R_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$$\frac{1}{3} R_3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/3 & -1 & 1/3 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + R_3$$
$$R_2 \rightarrow R_2 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1/3 & 0 & 1/3 \\ -7/3 & 2 & -2/3 \\ 5/3 & -1 & 1/3 \end{bmatrix}$$

4 (a) $f(x) = 4x^3 + x^2 - 4x - 1$ is Cont. in $[-1, 1]$

$f(x)$ is diff. in $(-1, 1)$

$f(-1) = f(1)$ by Rolle's Th., we get

$c = -2/3, \quad c = 1/2.$

4 (b) $y = (x^2-1)^n$ ——— (1)

$y_1 = n(x^2-1)^{n-1} 2x$

$(x^2-1) y_1 = 2nxy$ ——— (2)

Again differentiating

$(x^2-1) y_2 + 2xy_1 = 2n(xy_1 + y)$

$\Rightarrow (x^2-1) y_2 + 2(1-n)xy_1 - 2ny = 0$

Applying Leibnitz Th.

$(x^2-1) y_{n+2} + (2nx + 2x - 2nx) y_{n+1} + (n^2 - n + 2n - 2n^2 - 2n) y_n = 0$

$\Rightarrow (x^2-1) y_{n+2} + 2xy_{n+1} - n(n+1) y_n = 0.$

5(a) $\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - \left[y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2} \left(-\frac{x}{y^2}\right) + \left(\tan^{-1}\frac{x}{y}\right) 2y \right]$ (8)

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}.$$ (On simplification)

5(b) $$\operatorname{Cosec} u = \left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$$

$$n = 1/12$$

$$\phi(u) = n\frac{F(u)}{F'(u)} = \frac{1}{12} \frac{\operatorname{Cosec} u}{(-\operatorname{cosec} u \cot u)}$$

$$= -\frac{1}{12} \tan u$$

$$\phi'(u) = \frac{1}{12} \sec^2 u$$

by Euler's Th,

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \phi(u)[\phi'(u) - 1]$$

$$= -\frac{1}{12} \tan u \left[\frac{1}{12}\sec^2 u - 1\right]$$

$$= \frac{\tan u}{144}\left[13 + \tan^2 u\right]$$