Assignment (Unit 3)

(1) If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$, show that
$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

(2) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that
$$\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]^2 u = \frac{-9}{(x+y+z)^2}$$

(3) If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

(4) If $u = f(r)$ where $r^2 = x^2 + y^2$, prove that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

(5) If $z = \log(e^x + e^y)$, show that $\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$

(6) If $u = e^{xyz}$, prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)u$

(7) If $u = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(8) If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

(9) If $u = \log\sqrt{x^2 + y^2 + z^2}$, show that $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$

(10) If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, find the value of $\frac{\partial^2 u}{\partial x \partial y}$

(11) Find the value of n so that the equation $V = r^n(3\cos^2\theta - 1)$ satisfies the relation $\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial V}{\partial \theta}\right] = 0$

(12) Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the function $u = \log\left(\frac{x^2+y^2}{xy}\right)$

(13) Verify Euler's theorem for the following functions:

(a) $u = \frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$ (b) $u = \log\left(\frac{x^2+y^2}{xy}\right)$ (c) $u = \frac{x(x^4 - y^4)}{x^4 + y^4}$

(d) $u = \log\left(\frac{x^4 + y^4}{x + y}\right)$

(14) If $u = f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(15) If $u = x^3 + y^3 + z^3 + 3xyz$, show that $xu_x + yu_y + zu_z = 3u$

(16) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(17) If $u = xy f\left(\frac{x}{y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

(18) If $u = (x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})$, apply Euler's theorem to find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

(19) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

(20) If $\log u = \frac{x^3+y^3}{3x+4y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$

(21) If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

(22) If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

(23) If $u = \cos^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, show that $xu_x + yu_y + zu_z = -2\cot u$

(24) If $u = \sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$, prove that $xu_x + yu_y = \frac{5}{2}\tan u$

(25) If $u = \sin^{-1}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$, show that $xu_x + yu_y = -\frac{1}{12}\tan u$

(26) If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, show that $xu_x + yu_y + zu_z + 3\tan u = 0$

(27) If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

(28) If $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

(29) If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

(30) If $u = x\sin^{-1}\left(\frac{x}{y}\right) + y\sin^{-1}\left(\frac{y}{x}\right)$, evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

(31) If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u)$$

(32) State and prove Euler's theorem for partial differentiation of a homogeneous function $f(x,y)$.

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(33) Expand $\log(1+x)$ in powers of x. Then find series for $\log_e\left(\frac{1+x}{1-x}\right)$ and hence determine the value of $\log_e\left(\frac{11}{9}\right)$ upto five places of decimal.

(34) Show that $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)$

(35) Expand $e^{2x}\sin x$ in ascending powers of x up to x^5.

(36) Expand $e^{ax}\sin by$ in powers of x and y as far as the terms of third degree.

(37) Expand $e^{x}\cos y$ at $(1, \frac{\pi}{4})$

(38) Expand $\sin xy$ in powers of $(x-1)$ and $(y-\frac{\pi}{2})$ up to second degree terms.

(39) Expand $(x^2y+\sin y+e^x)$ in powers of $(x-1)$ and $(y-\pi)$

(40) Expand x^y in powers of $(x-1)$ and $(y-1)$ upto the third degree terms and hence evaluate $(1\cdot1)^{1\cdot02}$.

(41) Expand y^x about $(1,1)$ upto 2nd degree terms and hence evaluate $(1\cdot02)^{1\cdot03}$.

(42) Expand $e^x\tan^{-1}y$ in powers of $(x-1)$ and $(y-1)$ upto two terms of degree 2.

(43) Find the first six terms of the expansion of the function $e^x\log(1+y)$ in a Taylor series in the neighbourhood of the point $(0,0)$.

(44) Expand $\tan^{-1}\frac{y}{x}$ in the neighbourhood of $(1,1)$ upto and inclusive of second degree terms. Hence compute $f(1\cdot1, 0\cdot9)$ approximately.

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(45) If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta$

(46) If $u=x+2y+z$, $v=x+2y+3z$, $w=2x+3y+5z$, compute the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

(47) If $u=xyz$, $v=xy+yz+zx$, $w=x+y+z$, then compute the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

(48) If $y_1=\frac{x_2x_3}{x_1}$, $y_2=\frac{x_1x_3}{x_2}$, $y_3=\frac{x_1x_2}{x_3}$, show that $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}=4$

(49) If $u=x(1-r^2)^{-\frac{1}{2}}$, $v=y(1-r^2)^{-\frac{1}{2}}$, $w=z(1-r^2)^{-\frac{1}{2}}$, where $r^2=x^2+y^2+z^2$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)}=(1-r^2)^{-5/2}$

(50) If $x=u(1+v)$, $y=v(1+u)$, find $\frac{\partial(x,y)}{\partial(u,v)}$