ELEMENTS OF MECHANICAL ENGINEERING

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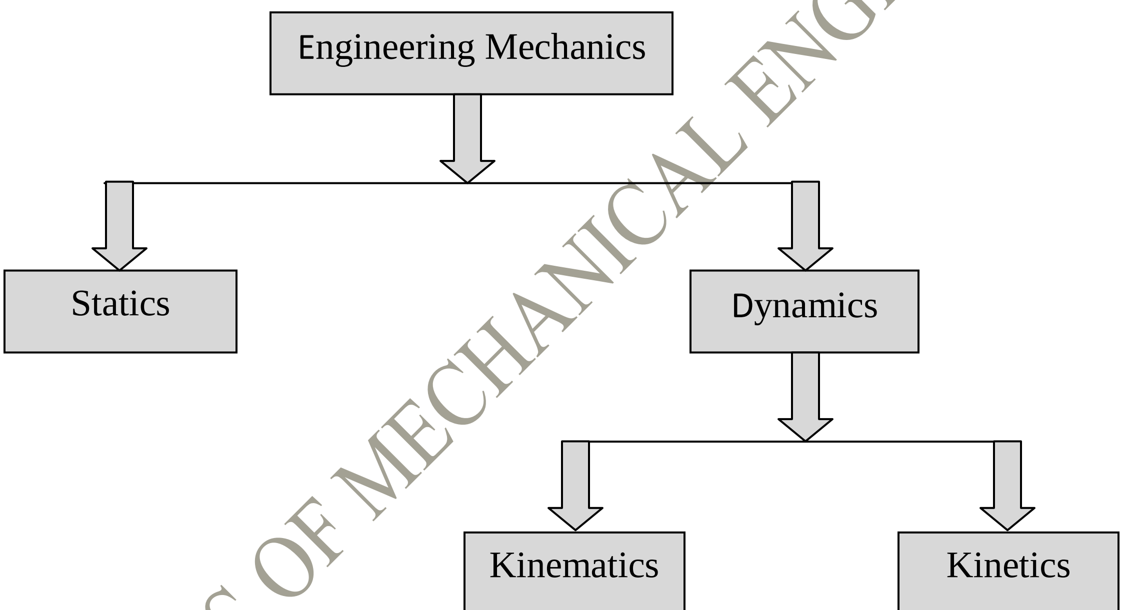
UNIT –I

Introduction:- Engineering mechanics is the branch of science which deals with the behavior of a body, when the body is at rest or in motion.

OR

Mechanics is a branch of science which deals with the action of forces on bodies.

Classification of engineering mechanics are shown in below:



Engineering mechanics may be divided into statics and dynamics.

(1) Statics:- The branch of science, which deals with the study of a body when the body is at rest is known as statics.

(2) Dynamics:- The branch of science which deals with the study of a body when the body is in motion is known as dynamics.

Dynamics is divided into *kinematics* and *kinetics*.

(i) Kinematics:- the study of a body in motion, when the force which cause the motion are not considered, is called kinematics.

(ii) **Kinetics:-** The study of body in motion, when the forces are also considered, is called kinetics.

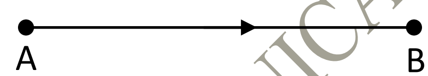
Rigid and Resistant(Elastics) body:-

- ➔ A body is said to be rigid if it is does not undergo deformation under the action of forces.
- ➔ A body which is capable to withstand forces acting on it with deformation but comes to its original position/condition after the removal of forces on the body, is known as Elastics or resistant body.

Scalar and Vector quantity:-

All physical quantities are divided into vector and scalar quantity.

Vector Quantity:- A quantity which is completely specified by magnitude and direction is known as vector quantity. Examples: velocity, acceleration, Force, momentum, etc.



The length of straight line AB represented by magnitude and arrow represent the direction of the vector.($\overrightarrow{AB}$ it means acting from A to B).

Scalar Quantity :- A quantity, which is completely specified by magnitude only, is known as scalar quantity. Example: mass, length, time, temp. volume etc.

Mass:- The quantity of the matter possessed by a body is called mass. The mass of a body will not change unless the body is damaged and part of it is physically separated. When a body is taken out in a spacecraft, the mass will not change but its weight may change due to change in gravitational force. Even the body may become weightless when gravitational force vanishes but the mass remain the same.

Particle:-

A particle may be defined as an object which has only mass and no size. Such a body cannot exist theoretically. However in dealing with problems involving distances considerably larger compared to the size of the body, the body may be treated as particle, without sacrificing accuracy. Examples of such situations are

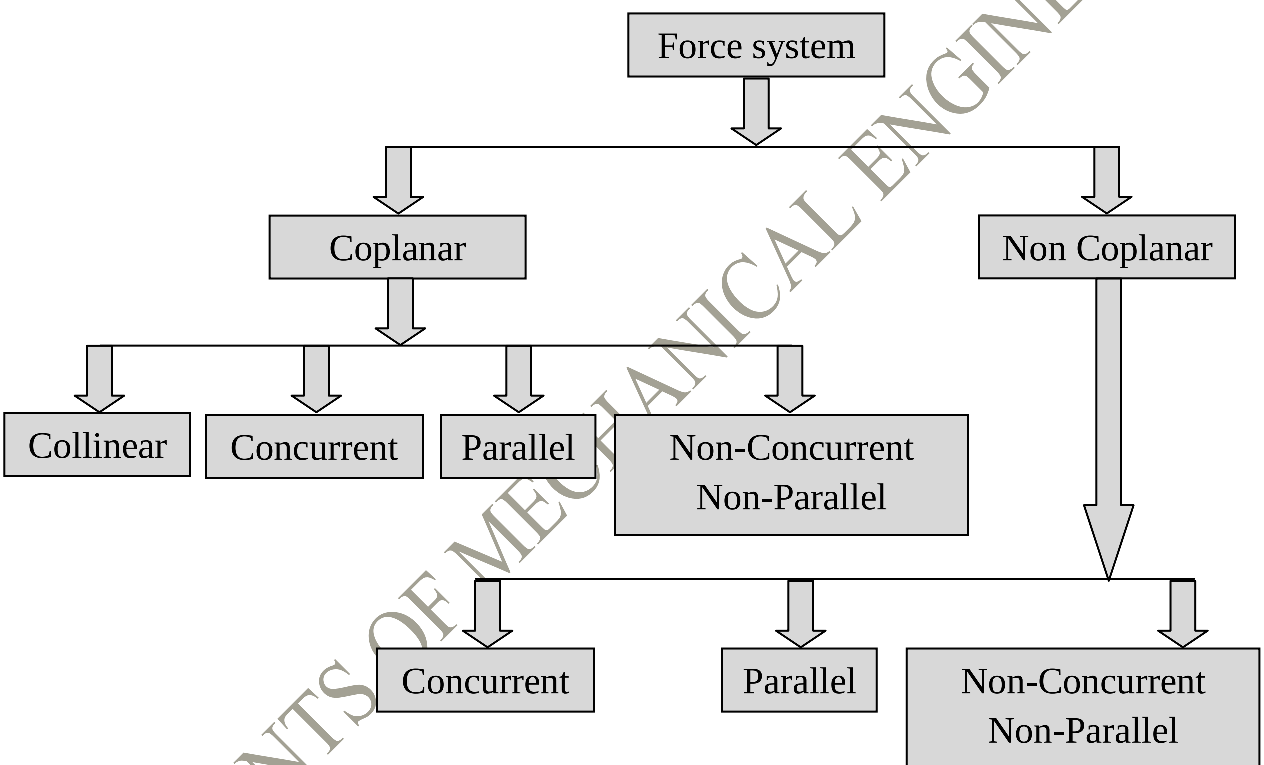
- ➔ A bomber aero plane is a particle for a gunner operating from the ground.
- ➔ A ship in mid sea is a particle in the study of its relative motion from a control tower.

- ➔ In the study of movement of the earth in celestial sphere, earth is treated as a particle.

Weight:-Weight is the force with which the system is attracted towards the centre of earth.

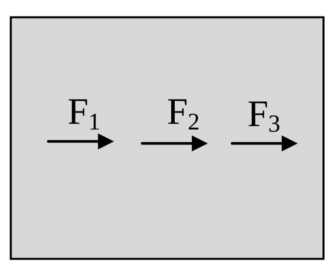
Force:- force is an external agent which tends to change the speed or direction of a system.

- ➔ For representing the force acting on the body, *the magnitude of the force*, *point of action* and *direction of its action*.

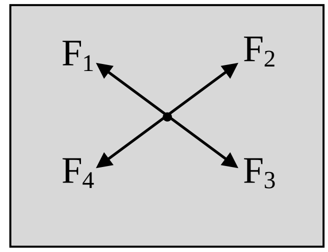


Coplanar Force System:-

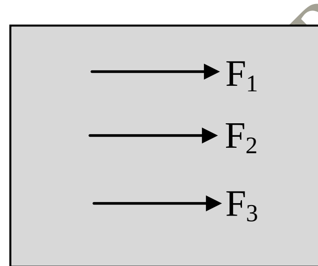
(1) Coplanar-collinear force system:-



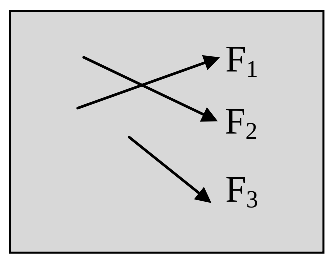
(2) Coplanar- concurrent force system:-



(3) Coplanar- parallel force system:-

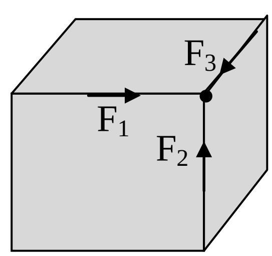


(4) Coplanar non-concurrent and non-parallel force system:-

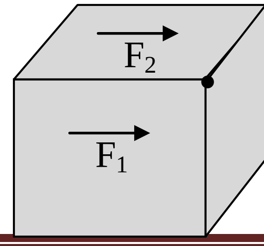


<u>**Non Coplanar Force System:-**</u>

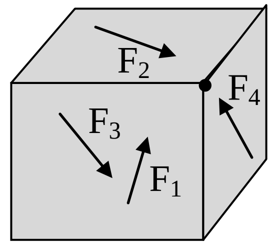
(1) Non-coplanar, concurrent force system:-



(2) Non-coplanar, parallel force system:-



(3) Non-coplanar, non concurrent nor parallel force system:-

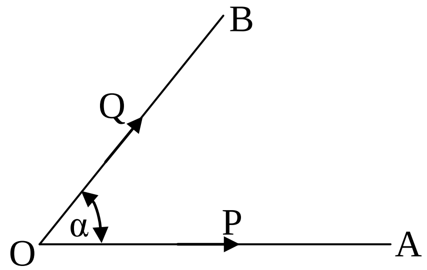


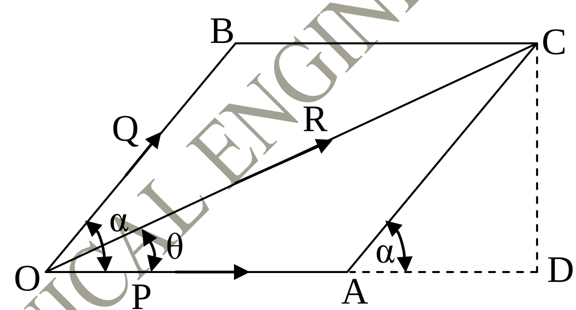
System of Forces		
Coplanar Force System		
1	Coplanar collinear forces	Line of action of all the forces act along the same line.
2	Coplanar concurrent forces	Line of action of all forces pass through a single point and forces lie in the same plane.
3	Coplanar parallel forces	All forces are parallel to each other and lie in a single plane.
4	Coplanar non-concurrent & non parallel forces	All forces do not meet at a point and not parallel to each other but lie in a single plane.
Non Coplanar Force System		
1	Non-coplanar concurrent forces	All forces do not lie in the same plane, but their lines of action pass through a single point.
2	Non-coplanar parallel forces	All the forces are parallel to each other, but not in same plane.
3	Non-coplanar non-concurrent & non parallel forces	All forces do not lie in the same plane and their lines of action do not pass through a single point and not parallel to each other.

Law Of Parallelogram Of Forces :-

The law of parallelogram of forces is used to determine the resultant of two forces at a point in a plane. It states, *"If two forces, acting at a point be represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing that point."*

Now consider a parallelogram OABC as shown in figure. Let sides OA and OB represent the force P and Q acting at point O. Then resultant of these forces R is represented by diagonal OC.





α = angle b/w force P and Q

θ = angle made by R with force P

Now ∠AOB = ∠DAC = α and OA=BC=P, OB=AC=Q

In Δ ACD-

$$AD = AC\cos\alpha = Q\cos\alpha$$
$$CD = AC\sin\alpha = Q\sin\alpha$$

Now consider Δ OCD-

$$OC^2 = OD^2 + CD^2$$
$$R^2 = (OA + AD)^2 + CD^2$$
$$= (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$$
$$= P^2 + Q^2\cos^2\alpha + 2PQ\cos\alpha + Q^2\sin^2\alpha$$
$$= P^2 + Q^2(\cos^2\alpha + \sin^2\alpha)^2 + 2PQ\cos\alpha$$
$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

Magnitude of resultant $R = \sqrt{(P^2 + Q^2 + 2PQ\cos\alpha)}$

Direction of resultant-

$$\tan\theta = \frac{CD}{OD} = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

$$\theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P + Q\cos\alpha}\right)$$

Case:I

If $\theta=90^0$ then cos90 = 0, sin90 = 1

$$R = \sqrt{(P^2 + Q^2)} \quad \text{and } \theta = \tan^{-1}\left(\frac{Q}{P}\right)$$

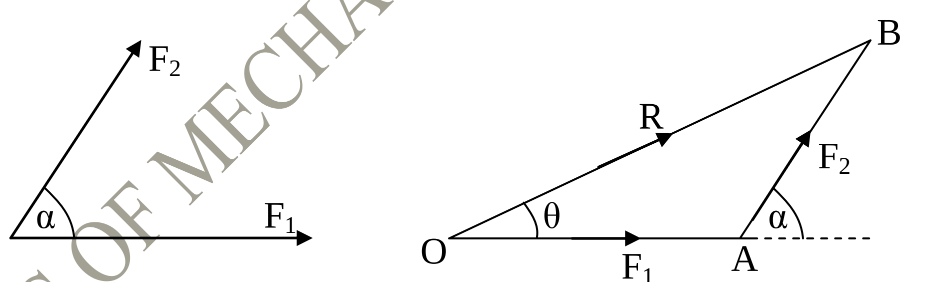
Case:II

If $\theta=0^0$ then cos0 = 1, sin0 = 0

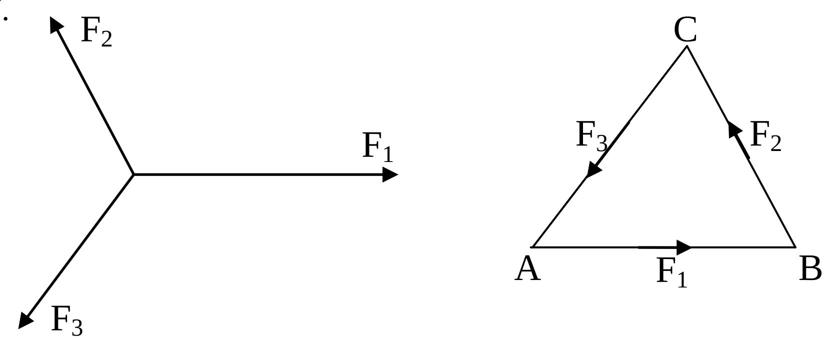
$$R = \sqrt{(P^2 + Q^2 + 2PQ)} = P + Q \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{\theta}{P + Q}\right) = 0$$

Triangle Law Of Forces:-

(1) As per this law, *"If two forces F_1 and F_2 acting at a point are represented by two side of a triangle then the third side will represent the resultant of two forces in the direction and magnitude"*.



(2) As per this law, *"If three forces acting at a point be represented in magnitude and direction by three side of triangle, taken in order, they will be in equilibrium"*.

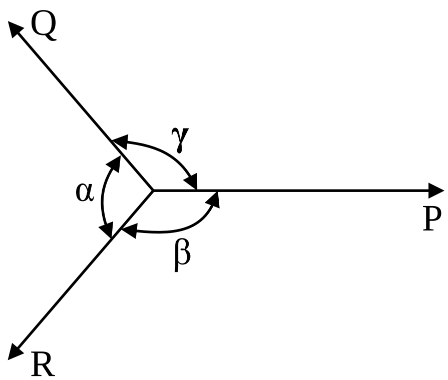


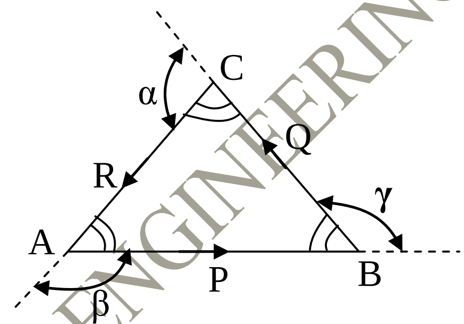
Lami's Theorem:-

If a particle remains in equilibrium under the action of three coplanar concurrent forces, then these forces are proportional to the sine of the angle between the other two forces.

Proof Of Lami's Theorem-

Let P, Q and R are three forces acting at a point O and α, β, γ are the angle b/w them as shown in figure.





Let α = angle between force Q and R

β = angle between force P and R

γ = angle between force P and Q

Then according to the Lami's theorem-

P ∝ sine of angle between Q and R ∝ sinα

$$\therefore \quad \frac{P}{\sin\alpha} = \text{constant}$$

Similarly-

$$\frac{Q}{\sin\beta} = \text{constant } and \ \frac{R}{\sin\gamma} = \text{constant}$$

$$\text{OR} \quad \frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Now apply sine rule in triangle ABC-

$$\frac{P}{\sin(180-\alpha)} = \frac{Q}{\sin(180-\beta)} = \frac{R}{\sin(180-\gamma)}$$

This can also be written as-

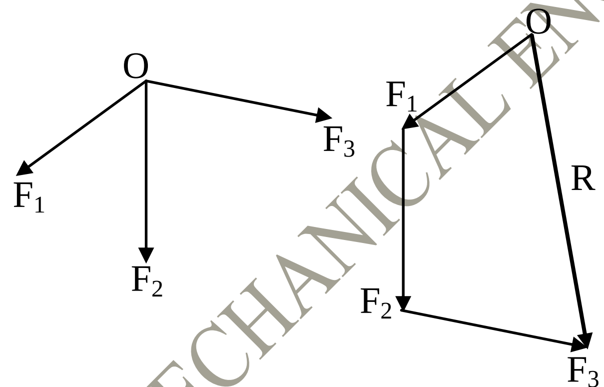
$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} \dots\dots\dots\dots\dots\dots\dots\dots\dots (B)$$

Equation (A) is same as equation (B).

Polygon law of forces:-

When the forces acting on a particle are more than two, the triangle can be extended to polygon law.

"If a number of concurrent forces acting on a body be represented in magnitude and direction by the sides of the polygon, taken in order, then these resultant can represented by closing side of the polygon in magnitude and direction in the opposite order".

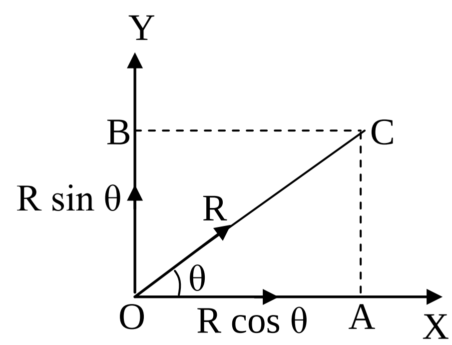


Resolution of a Force :-

Resolution of force means " finding the component of a given force in two given direction."

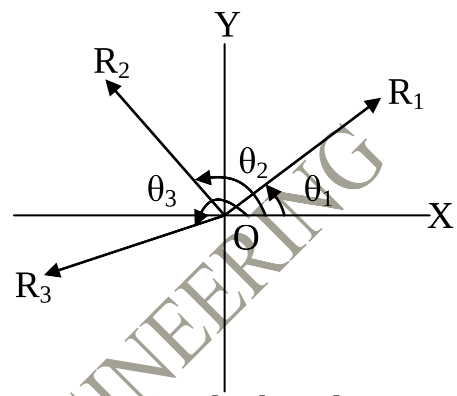
Let a given force be R which make angle θ with X-axis. It is required to find the component of force R along X-axis and Y-axis.

Component of R along X-axis = R cos θ
Component of R along Y-axis = R sin θ
Hence, the resolution of forces is the process of finding component, of forces in specified directions.



Resolution Of A No. Of Coplanar Forces:-

Let a no. of coplanar forces R_1, R_2, R_3…………are acting at a point as shown in figure.



Let θ_1= Angle made by R_1 with X-axis.

θ_2= Angle made by R_2 with X-axis

θ_3= Angle made by R_3 with X-axis

H= Resultant component of all forces along X-axis.

V= Resultant component of all forces along Y-axis

R= Resultant of all forces

θ = angle made by resultant with X-axis

Each force can be resolved into two components, one along X-axis and other along Y-axis

Component of R_1 along X-axis = $R_1\cos\theta_1$

Component of R_1 along Y-axis = $R_1\sin\theta_1$

Similarly, the components of R_2 and R_3 along X-axis and Y-axis are ($R_2\cos\theta_2$, $R_2\sin\theta_2$) and ($R_3\cos\theta_3$, $R_3\sin\theta_3$) respectively

Resultant component along X-axis

= sum of component of all forces along X-axis.

$\therefore H = R_1\cos\theta_1 + R_1\cos\theta_2 + R_3\cos\theta_3 +$ ----------------

Resultant component along Y-axis

= Sum of components of all component along Y – axis.

$\therefore V = R_1\sin\theta_1 + R_2\sin\theta_2 + R_3\sin\theta_3 +$ -------------------

Then resultant of all the forces,

$$R = \sqrt{H^2 + V^2}$$

The angle made by R with X – axis is given by,

$$\tan\theta = \frac{V}{H}$$

Laws Of Mechanics:-

The following are the fundamental laws of mechanics:

(1) Newton's first law
(2) Newton's second law
(3) Newton's third law
(4) Newton's law of gravitation, and
(5) Law of transmissibility of forces

Newton's First Law:-

"It states that every body continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external agency acting on it". This leads to the definition of force as the external agency which changes or tends to change the state of rest or uniform linear motion of the body.

Newton's Second Law:-

"It states that the rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction of the force acting on it". Thus according to this law,

Force $\propto$ rate of change of momentum.

But momentum = mass × velocity

As mass do not change,

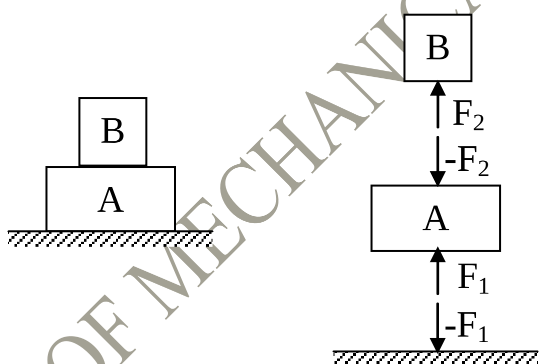
Force $\propto$ mass × rate of change of velocity

i.e., Force $\propto$ mass × acceleration

$F \propto m \times a$

Newton's Third Law:-

It states that for every action there is an equal and opposite reaction.



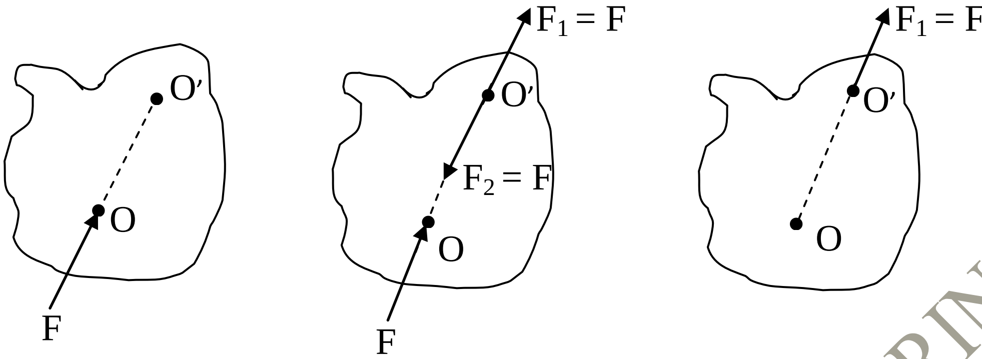
Newton's Law Of Gravitation:-

Everybody attracts the other body. The force of attraction between any two bodies is directly proportional to their masses and inversely proportional to the square of the distance between them. According to this law the force of attraction between the bodies of mass m_1 and mass m_2 at a distance d .

$$F = G\frac{m_1 m_2}{d^2}$$

The Principle of Transmissibility of Forces :-

It states that if a force, acting at a point on a rigid body, is shifted to any other point which to on the line of action of the force, the external effect of the force on the body remains unchanged.



Example-

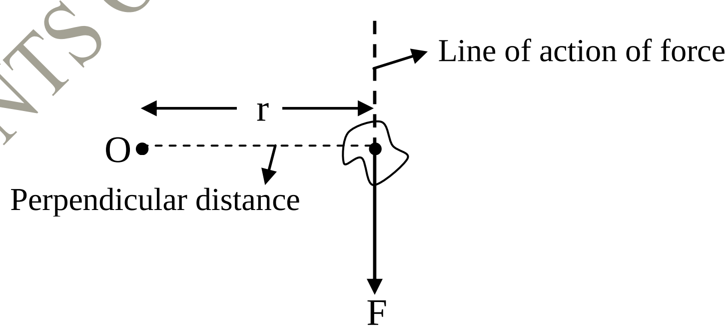
Consider a force F acting at point O on a rigid body . On this rigid body, "there is another point O' on the line of action o force F . Suppose at this point O' two equal and opposite forces F_1 and F_2 are applied. The force F and F_2 being equal and opposite, will cancel each other, leaving a force F_1 at pt. O'. But force F_1 is equal to F.

The original force F acting at point O, has been transferred to point O' which is along the line of action of F without changing the effect of forces on rigid body.

Hence any force acting at a point on a rigid body can be transmitted to act any other point along its line of action without changing its effect on rigid body. This proves the principle of transmissibility of a force.

Moment of a force:-

The product of a force and the perpendicular distance of the line of action of the force from a point is known as moment of the force about that point.



Let F = A force acting on a body

r = perpendicular distance from the point O on the line of action of force F.

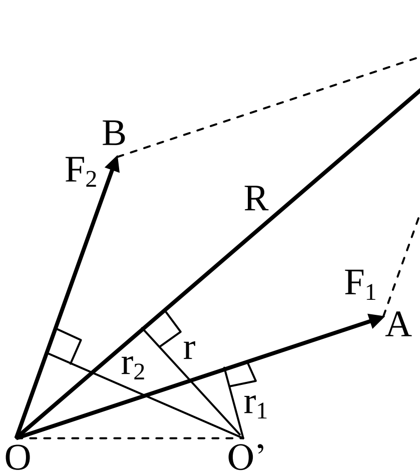
then moment (M) of the force about O is given by,

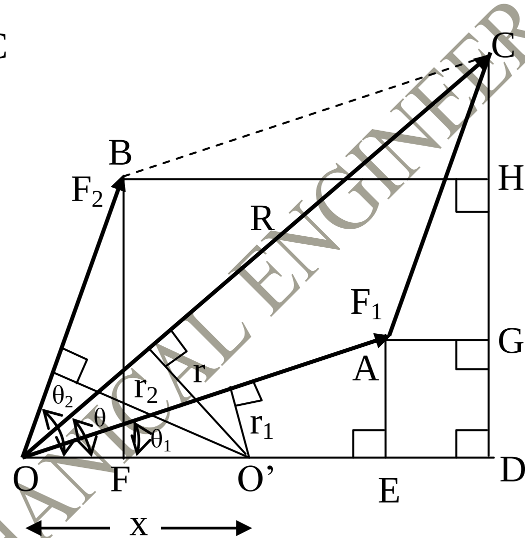
$$M = F \times r$$

The tendency of this moment is to rotate the body in clockwise direction about O. Hence this moment is called *clockwise moment.* If the tendency of moment is to rotate the body in anti-clockwise direction, then moment is known as *anti-clockwise moment.*

<u>Varignon's Theorem(Or Principle Of Moments):-</u>

Varignon's theorem states that "*the moment of a force about any point is equal to algebraic sum of the moments of its components about that point*".





Proof Of Varignon's Theorem-

- ➔ Fig.I shows two forces F_1 and F_2 at a point O are represented by in magnitude and direction along OA and OB. Resultant of F_1 and F_2 are R which is represented by OC.
- ➔ Let a point O' in the plane about which moments of F_1 and F_2 and R are to be determined.
- ➔ Draw perpendiculars on OA, OB and OC from point O'.
- ➔ r_1, r_2 and r are the perpendicular distance between F_1 and O', F_2 and O' and R and O' respectively.

Then according to the varignon's theorem-

Moment of R about O' is equal to the algebraic sum of moments of F_1 and F_2 about O'.

$$R \times r = F_1 \times r_1 + F_2 \times r_2$$

Now consider fig. II-

θ_1 = angle made by F_1 with OD

θ_2= angle made by F2 with OD

θ= angle made by R with OD

From fig. OA = BC and OA // BC hence the projection of OA and BC on the same vertical line CD will be equal i.e. GD = CH then we have-

$F_1 \sin\theta_1 = AE = GD = CH$

$F_1 \cos\theta_1 = OE$

$F_2 \sin\theta_2 = FB = HD$

$F_2 \cos\theta_2 = OF = ED$

$R \sin\theta = CD$ and $R\cos\theta = OD$

Let OO' = x

Then $x \sin\theta_1 = r_1$, $x \sin\theta = r$ and $x \sin\theta_2 = r_2$

Now moment of R about O'-

= R × (perpendicular distance b/w O' and R)

$= R \times r = R\,(x \sin\theta) = R \sin\theta \times x = CD \times x$

$= (CH + HD) \times x = (F_1 \sin\theta_1 + F_2 \sin\theta_2) \times x$

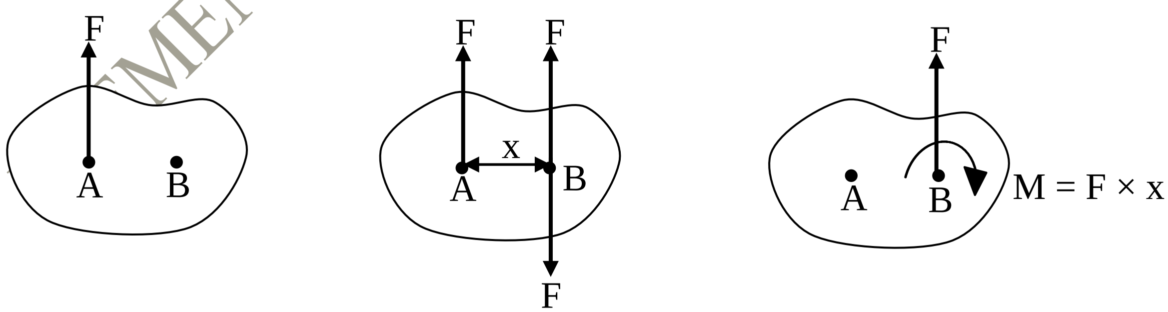
$= F_1 \sin\theta_1\, x + F_2 \sin\theta_2\, x = F_1 r_1 + F_2 r_2$

= moment of F_1 about O' + moment of F_2 about O'

$R \times r = F_1 r_1 + F_2 r_2$

Transfer Of A Force To Parallel Position:-

A given force f applied to a body at any point A can always be replaced by equal and parallel force applied at another point B together with a couple which will be equivalent to the original force.



Principle Of Equilibrium:-

The principle of equilibrium states that, a stationary body which is subjected to coplanar forces (concurrent or parallel) will be equilibrium if the algebraic sum of all the external forces is zero and also the algebraic sum of moments of all external forces about any point in their plane is zero.

Mathematically,

$$\sum F = 0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\sum M = 0 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Equation (1) is known as force law of equilibrium.

Equation (2) is known as moment law of equilibrium

(1) Equation Of Equilibrium For Coplanar Concurrent Forces System:-

$$\sum F_x = 0$$

$$\text{and } \sum F_y = 0$$

where $\sum F_x$ = Algebraic sum of all horizontal components.

and $\sum F_y$ = Algebraic sum of all vertical components.

(2) Equation Of Equilibrium For Coplanar Non-concurrent Forces System:-

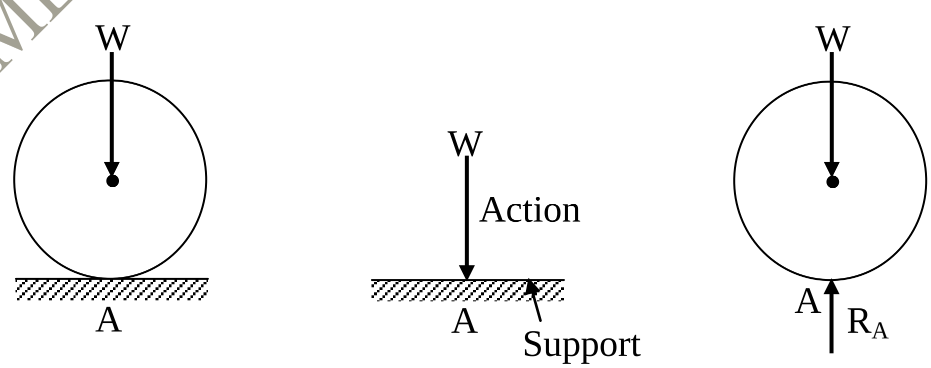
$$\sum F_x = 0$$

$$\sum F_y = 0$$

and $\sum M = 0$ (moment about any point is equal to zero)

Action And Reaction:-

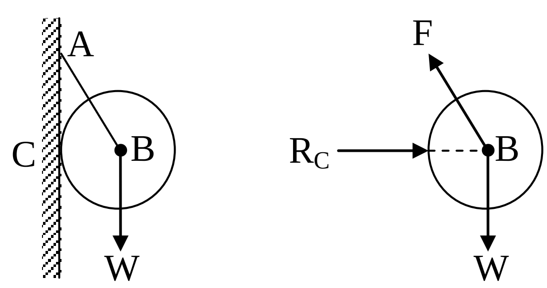
According to Newton's third law, for every action there is equal and opposite reaction. Consider a ball on a horizontal surface-



Free Body diagram:-

The equilibrium of the bodies which are placed on the supports can be considered if we removed the supports and replace them by reactions which they exert on the body.

Now we explain with a example-



(a) (b)

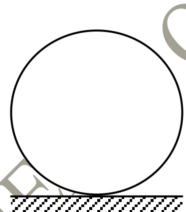
In above figure, if we remove the supporting surface and replace it by reaction R_C that the surface exerts on the balls.

Figure (b) is the free body diagram of the figure (a).

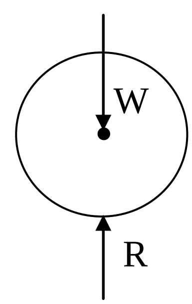
➔ In figure (b), in which the ball is completely isolated from its support and in which all forces acting on the ball are shown by vectors, is known as free body diagram.

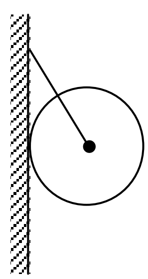
➔Hence to draw the free body diagram of a body we remove all the supports (like wall, floor, hinge of any other body) and replace them by reaction.

Free Body Diagram For A Few Cases:-

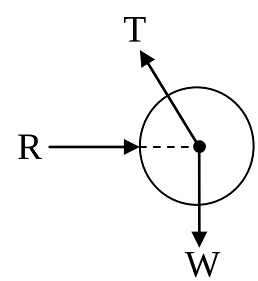


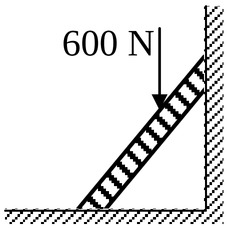
F.B.D. For Ball



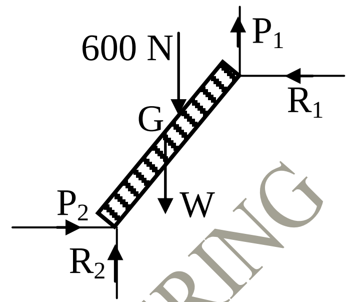


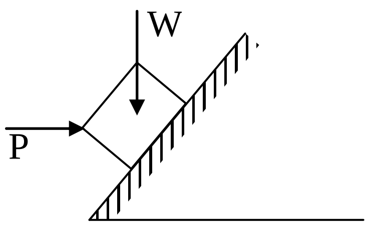
F.B.D. For Ball



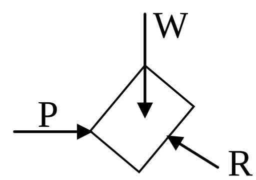


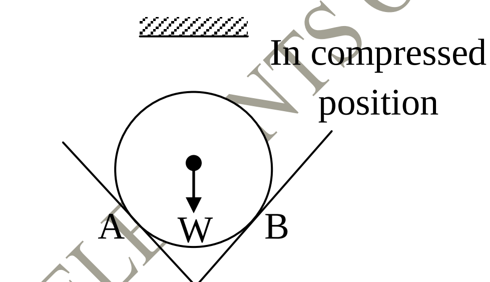
F.B.D. For Ladder



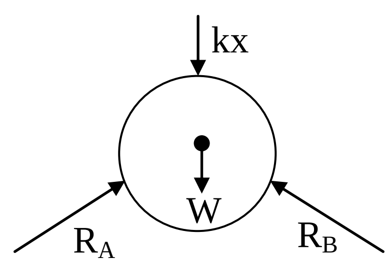


F.B.D. For Block



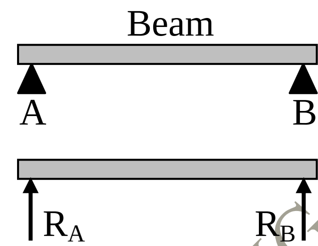


F.B.D. For Sphere

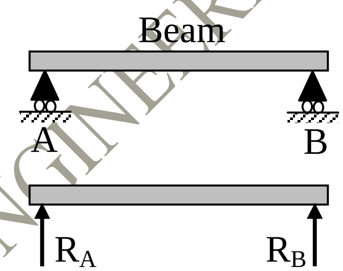


Types Of Supports:- Various types of supports and reactions developed are listed below-

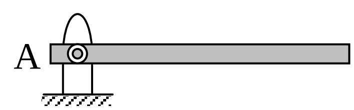
(1) Simple Support Or Knife Edge Support:-
If a beam rest simply on a support it is called a simply support. In this only support reaction acts at right angle to the support.



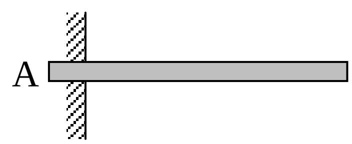
(2) Roller Support:-
In this case , beam end is supported on roller. In this case only normal reaction act on support since roller can be treated as frictionless.



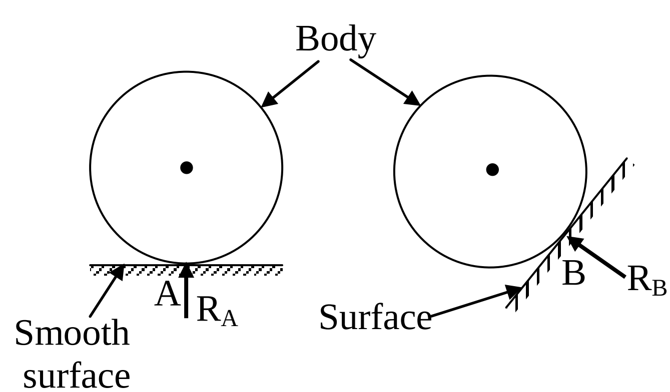
(3) Hinged Support:-
At a hinged end, a beam can not move in any direction. However, it can rotate about the support. In this case reaction R can be split into its horizontal and vertical component for the purpose of analysis.



(4) Fixed Support:-
At such support the beam end is not free to translate nor rotate. Translation is prevented by support reaction in any required direction and rotation is prevented by developing support moment M_A.



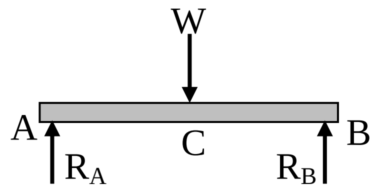
(5) Smooth Surface Support:-
Fig. shows a body in contact with a smooth surface. The reaction will always act normal to the support as shown in figure.



Types Of Loading:-

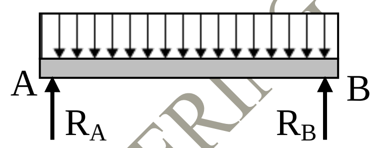
(1) Concentrated Loads:-

If a load is acting on a beam over a very small length (i.e. at a point) is called concentrated load.



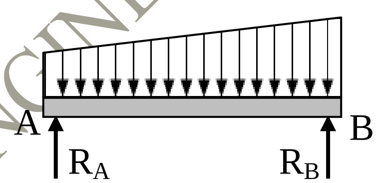
(2) Uniformly Distributed Load (UDL):-

If load is acting over considerable long distance and such load has got uniform intensity.



(3) Uniformly Varying Load (UVL):-

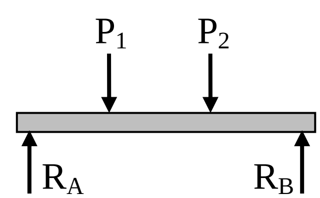
If load is acting over a length but its intensity is varying from 0 to w/unit length.



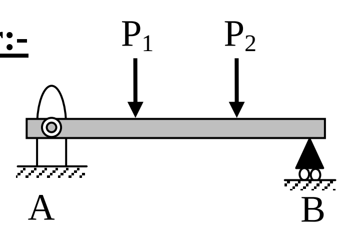
Types Of Beam:-

A beam may be defined as a structural element which has one dimensional considerably larger than other than two dimension and is supported at few point. It is usually loaded normal to its axis.

(1) Simply Supported Beam:-

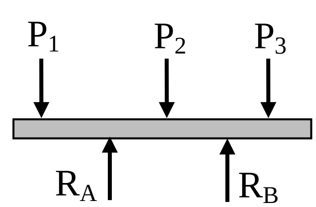


(2) Beam With One End Hinged And Other On Roller:-



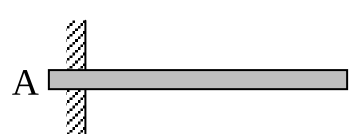
(3) Over Hanging Beam:-

If the beam is projecting beyond the support. It is called on over hanging beam.



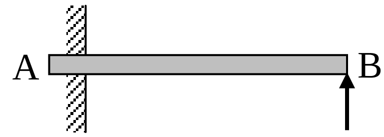
(4) Cantilever Beam:-

If a beam is fixed at one end and is free at the other end. It is called cantilever beam.

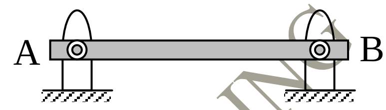


(5) Propped Cantilever Beam:-

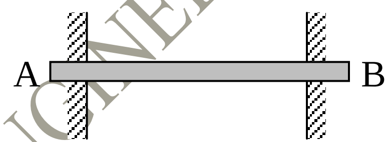
One end is fixed and other is simply supported.



(6) Both Ends Hinged:-

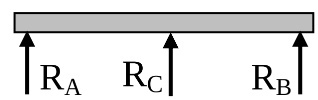


(7) Both Ends Fixed:-



(8) Continuous Beam:-

Beam is said to be continuous if it is supported at more than two points.



Note:- In the case of *simply supported beam, beams with one end hinged and other on roller, cantilever beams and over hanging beams*, it is possible to determine the reactions for given loading by using the equation of equilibrium only.

Determinate Beams:- The beam which can be analysed using only equilibrium equations are knows as statically *Determinate Beams*.

Indeterminate Beams:- Those beams which can not be analysed by equilibrium equation are known as *Indeterminate Beams*.

Centroid& Centre Of Gravity

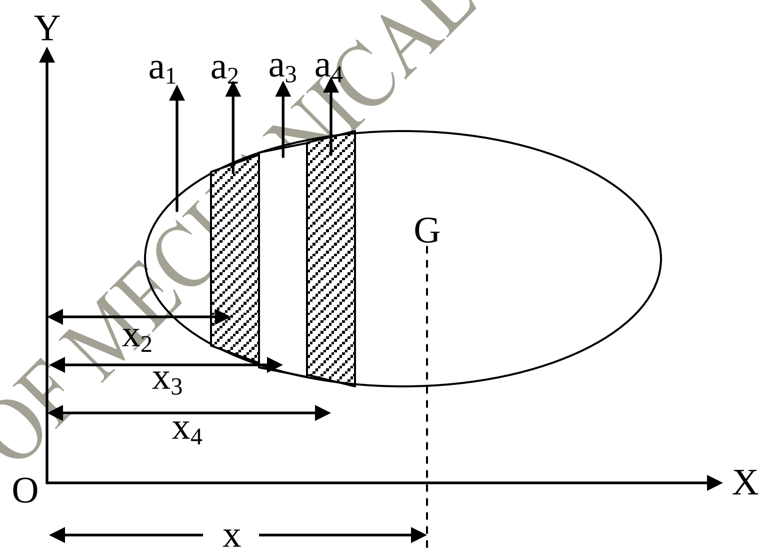
Centre Of Gravity: - Centre of gravity of a body is the point through which the whole weight of the body acts. A body is having only one centre of gravity (C.G.) for all position of the body.

Centroid: - The centroid or centre of area (G) is define as the point where the whole area of the figure is assumed to be concentrated.

- The centroid can be taken as quite analogous to centre of gravity when bodies have area only and not weight.

Centroid Of Area Of Plane Figures By The Method Of Moment :-

Consider a plane figure of total area A whose centre of area is to be determined.



Let this area A is composed of a number of small areas a_1, a_2, a_3, a_4,…………..,a_n

$$A = a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots + a_n$$

Let x_1 = The distance of the G of the area a_1 from axis OY

x_2 = The distance of the G of the area a_2 from axis OY

x_3 = The distance of the G of the area a_3 from axis OY

………………………………………………………..

The moments of all small areas about the axis OY = $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_nx_n$

Let G is the centroid of total area A whose distance from the axis OY is $\bar{x}$

The moment of total area about OY = $A\bar{x}$

The moments of all small areas about the axis OY must be equal to the moment of total area about the same axis OY. Hence

$$A\bar{x} = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots\ldots\ldots\ldots\ldots + a_nx_n$$

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + \cdots \ldots\ldots\ldots.. + a_nx_n}{A}$$

Where A = $a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots\ldots + a_n$

Similarly

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + \cdots \ldots\ldots\ldots.. + a_ny_n}{A}$$

Let y_1 = The distance of the G of the area a_1 from axis OX

y_2 = The distance of the G of the area a_2 from axis OX

y_3 = The distance of the G of the area a_3 from axis OX

…………………………………………………………..

Determination Of Centroid Of Simple Figure From First Principle :-

For simple figure like triangle, rectangle, semicircle etc. we can write general expression for the elemental area and its distance from an axis

$$\bar{x} = \frac{\int x^* \, dA}{\int dA}$$

$$\bar{y} = \frac{\int y^* \, dA}{\int dA}$$

Where $\int x^* \, dA = \sum x_i a_i$, $\int y^* \, dA = \sum y_i a_i$ and $\int dA = \sum a_i$

x^* = Distance of G of area dA from axis OY and

y^* = Distance of G of area dA from axis OX

Centroid Or Centre Of Gravity Of A Line :-

The centre of gravity of a line which may be straight or curve , is obtained by dividing the given line into a large number of small lengths as shown in figure.

The centre of gravity is obtained by

$$\bar{x} = \frac{\int x^* \, dL}{\int dL}$$

$$\bar{y} = \frac{\int y^* \, dL}{\int dL}$$

where x^* = Distance of C.G. of area dL from axis OY and

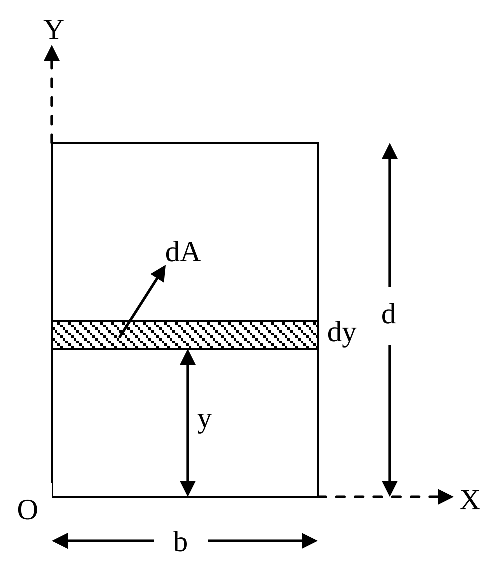
y^* = Distance of C.G. of area dL from axis OX

If the lines are straight, then the above equations are written as:

$$\bar{x} = \frac{L_1x_1 + L_2x_2 + L_3x_3 + \cdots \ldots \ldots \ldots .. L_nx_n}{L}$$

$$\bar{y} = \frac{L_1y_1 + L_2y_2 + L_3y_3 + \cdots \ldots \ldots \ldots .. L_ny_n}{L}$$

Centroid Of A Rectangle:-



Consider a rectangle of width b and depth d.

Consider a strip of depth dy at a distance y from OX.

Area of strip dA = dy.b

$$\bar{y} = \frac{\int y^* \, dA}{\int dA} = \frac{\int_0^d y.bdy}{A}$$

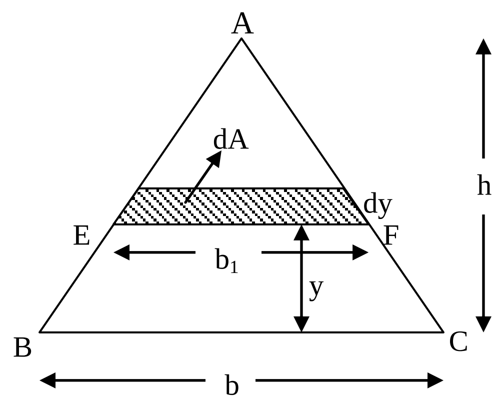
$$\bar{y} = \frac{b\left[\frac{y^2}{2}\right]_0^d}{bd} = \frac{d}{2}$$

$$\bar{y} = \frac{d}{2}$$

Similarly $\bar{x} = \frac{b}{2}$

❖ Hence centroid of rectangle $(\bar{x}, \bar{y}) = \left(\frac{b}{2}, \frac{d}{2}\right)$

Centroid Of A Triangle:-



Consider a triangle ABC of base b and height h.

Let us locate the distance of centroid from the base.

Consider a strip of depth dy at a distance y from base.

From similar Δ AEF and ΔABC-

$$\frac{b_1}{b} = \frac{(h-y)}{h}$$

$$b_1 = \left(\frac{h-y}{h}\right)b = \left(1 - \frac{y}{h}\right)b$$

Area of strip = dA = b_1.dy = $\left(1 - \frac{y}{h}\right)b.dy$

Area of triangle A = $\frac{1}{2}bh$

$$\bar{y} = \frac{\int y^* dA}{A}$$

Now $\int y^* dA = \int_0^h y.\left(1 - \frac{y}{h}\right)b.dy$

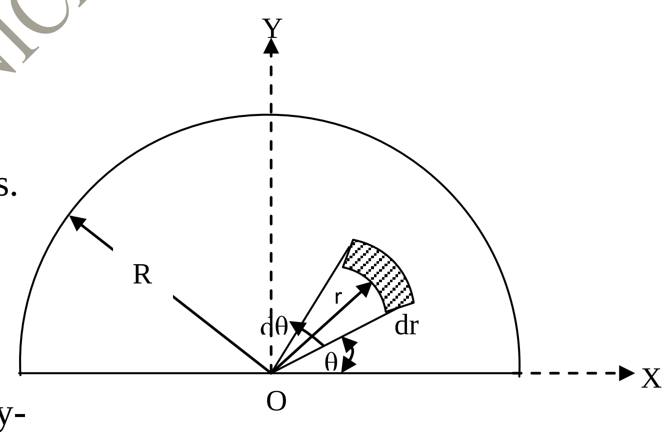
$$= \int_0^h \left(y - \frac{y^2}{h}\right) b.\,dy = b\left[\frac{y^2}{2} - \frac{y^3}{3h}\right]_0^h$$

$$= \frac{bh^2}{6}$$

$$\bar{y} = \frac{\int y^* dA}{A} = \frac{\frac{bh^2}{6}}{\frac{bh}{2}} = \left(\frac{h}{3}\right)$$

- Thus the centroid of a triangle is at a distance $\left(\frac{h}{3}\right)$ from the base or $\left(\frac{2h}{3}\right)$ from the apex of the triangle where h is the height of triangle.

Centroid Of A Semicircle:-



Consider a semicircle of radius R.

Due to symmetry centroid must lies on Y axis.

Let its distance from diametral axis be $\bar{y}$.

Area of element = rdθ.dr

Its moment about diametral axis X is given by-

$$= r d\theta\, dr.\, r\sin\theta$$

$$= r^2 \sin\theta\, d\theta\, dr$$

Total moment of area about diametral axis X-

$$\int_0^{\pi}\int_0^{R} r^2 \sin\theta\, d\theta\, dr = \int_0^{\pi} \left[\frac{r^3}{3}\right]_0^R \sin\theta\, d\theta$$

$$= \frac{R^3}{3}\int_0^{\pi} \sin\theta\, d\theta = \frac{R^3}{3}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{2R^3}{3}$$

Area of semicircle A= $\frac{1}{2}\pi R^2$

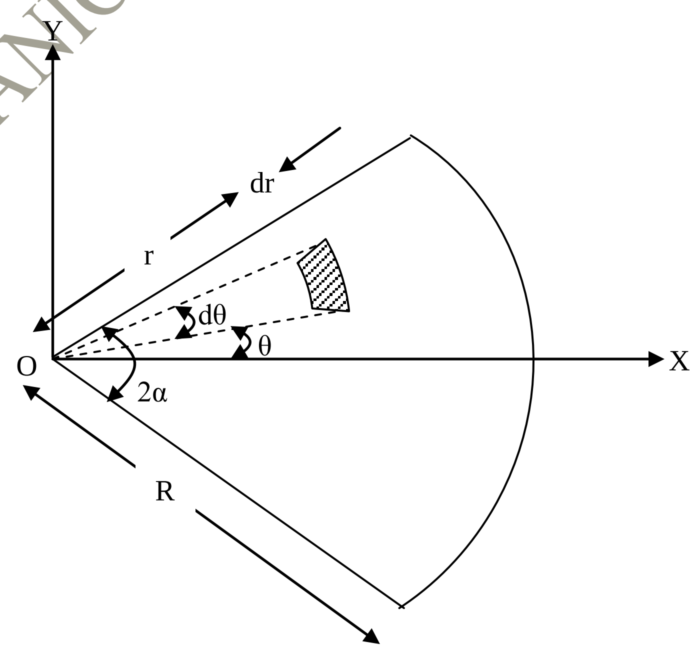
$$\bar{y} = \frac{\text{total moment of area}}{\text{total area}}$$

$$\bar{y} = \frac{\frac{2R^3}{3}}{\frac{1}{2}\pi R^2}$$

$$\bar{y} = \left(\frac{4R}{3\pi}\right)$$

❖ Thus the centroid of the circle is at a distance $\left(\frac{4R}{3\pi}\right)$ from the diametral axis.

Centroid Of Sector Of A Circle:-



Consider the sector of circle of angle 2α.

Due to symmetry centroid lies on X axis.

To find the its distance from the centre O

Consider the elemental area as shown

In figure-

Area of element = rdθ.dr

Its moment about Y axis-

$= r d\theta \; dr. \; r\cos\theta$

$= r^2 \cos\theta \; d\theta \; dr$

Total moment of area about Y axis-

$$\int_{-\alpha}^{\alpha}\int_{0}^{R} r^2\cos\theta\, d\theta\, dr = \int_{-\alpha}^{\alpha} \left[\frac{r^3}{3}\right]_0^R \cos\theta\, d\theta$$

$$= \frac{R^3}{3}\int_{-\alpha}^{+\alpha} \cos\theta\, d\theta = \frac{R^3}{3}[\sin\theta]_{-\alpha}^{+\alpha}$$

$$= \frac{2R^3}{3}\sin\alpha$$

Total area of the sector = $\int_0^R \int_{-\alpha}^{\alpha} r d\theta dr = R^2\alpha$

The distance of centroid from centre O

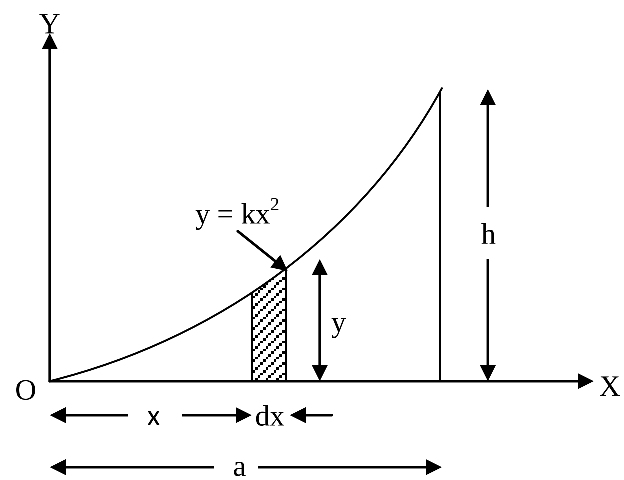
$$= \frac{\text{moment of area about Y axis}}{\text{total area of the sector}}$$

$$= \frac{\frac{2R^3}{3}\sin\alpha}{R^2\alpha}$$

$$\bar{x} = \left(\frac{2R}{3\alpha}\sin\alpha\right)$$

Centroid Of Parabolic Spandrel:-

Consider the parabolic spandrel as shown in figure.



Cut a element of width dx and height y at a distance x from O.

Area of the element = $kx^2.dx$

Moment of area about Y axis-

$$\int_0^a kx^2 dx.\, x = \frac{ka^4}{4}$$

Moment of area about X axis-

$$\int_0^a kx^2 dx.\frac{y}{2} = \int_0^a kx^2 dx.\frac{kx^2}{2}$$

$$= \int_0^a \frac{k^2x^4}{2}\ dx$$

$$= \frac{k^2a^5}{10}$$

Total area of the spandrel

$$= \int_0^a kx^2 . dx$$

$$= \frac{ka^3}{3}$$

$$\bar{x} = \frac{\text{moment of area about Y axis}}{\text{total area of the spandrel}}$$

$$\bar{x} = \frac{\frac{ka^4}{4}}{\frac{ka^3}{3}} = \frac{3a}{4}$$

$$\bar{y} = \frac{\text{moment of area about X axis}}{\text{total area of the spandrel}}$$

$$\bar{y} = \frac{\frac{k^2a^5}{10}}{\frac{ka^3}{3}} = \frac{3}{10}ka^2$$

At x=a, y=h then h=ka^2

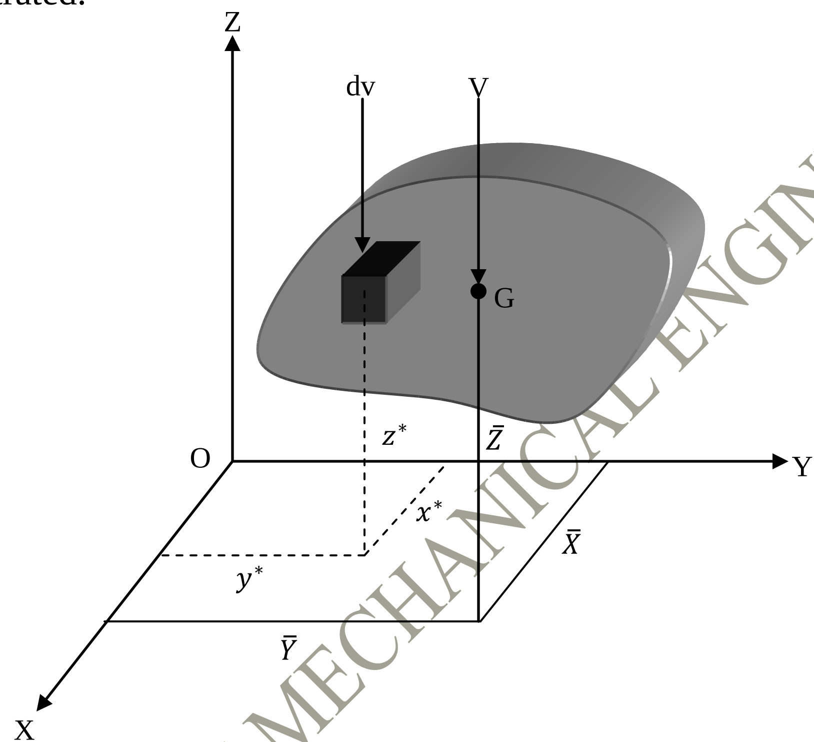
$$k = \frac{h}{a^2}$$

$$\bar{y} = \frac{3}{10}ka^2 = \frac{3}{10}\frac{h}{a^2}a^2 = \frac{3h}{10}$$

- Hence centroid spandrel $(\bar{x}, \bar{y}) = \left(\frac{3a}{4}, \frac{3h}{10}\right)$

Centroid Of Volume:-

Centroid of volume is the point at which the total volume of a body is assumed to be concentrated.



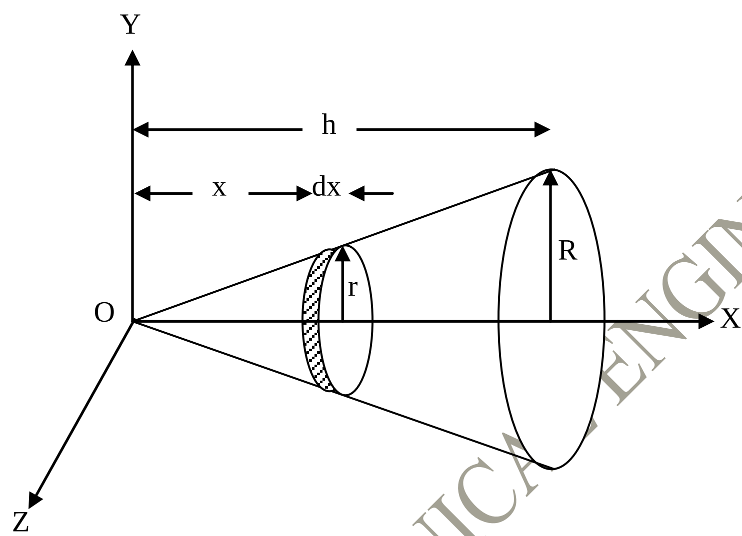
The centroid of the volume is obtained by-

$$\bar{x} = \frac{\int x^* \, dv}{\int dv}$$

$$\bar{y} = \frac{\int y^* \, dv}{\int dv}$$

$$\bar{z} = \frac{\int z^* \, dv}{\int dv}$$

Centroid Of The Cone:-



To find the $\bar{x}$, consider a small volume dv. For this take a small circular plate at a distance x from O. Let the thickness of the plate is dx and radius r.

From similar triangle- $\frac{R}{r} = \frac{h}{x}$

$$r = \frac{Rx}{h}$$

Volume of circular plate dv = $\pi r^2.dx$

$$dv = \frac{\pi R^2}{h^2} x^2 dx$$

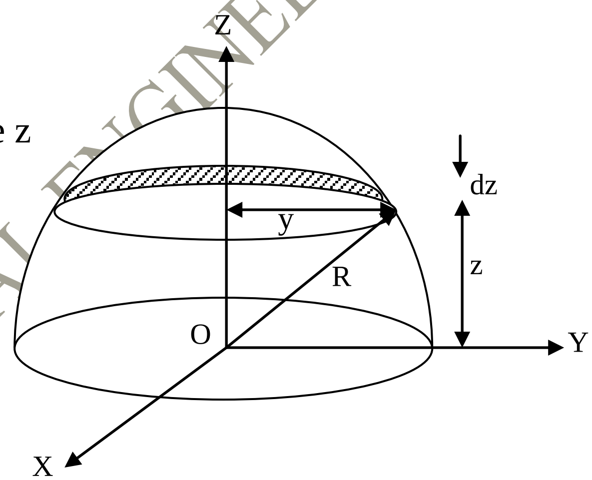
Centroid of the volume of the cone

$$\bar{x} = \frac{\int x^* \, dv}{\int dv} = \frac{\int_0^h x.\frac{\pi R^2}{h^2} x^2 \, dx}{\int_0^h \frac{\pi R^2}{h^2} x^2 \, dx}$$

$$\bar{x} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h} = \frac{3h}{4}$$

$$\bar{x} = \left(\frac{3h}{4}\right)$$

Centroid Of Hemisphere:-



Consider a hemisphere of centre O and radius R.

Cut a disc of radius y and thickness dz at a distance z from centre O.

volume of the circular disc $dv = \pi y^2 dz$

and $y^2 + z^2 = R^2$

$y^2 = R^2 - z^2$

$\therefore dv = \pi(R^2 - z^2)dz$

As in this case axis of symmetry is Z axis so $\bar{x}$ and $\bar{y}$ are zero.

$$\bar{z} = \frac{\int z^* dv}{\int dv} = \frac{\int_0^R z.\pi\left(R^2 - z^2\right)dz}{\int_0^R \pi(R^2 - z^2)dz}$$

$$= \frac{\pi\left[\frac{R^2 z^2}{2} - \frac{z^4}{4}\right]_0^R}{\pi\left[R^2 z - \frac{z^3}{3}\right]_0^R}$$

$$= \frac{\left[\frac{R^4}{2} - \frac{R^4}{4}\right]}{\left[R^3 - \frac{R^3}{3}\right]} = \frac{\frac{R^4}{4}}{\frac{2R^3}{3}}$$

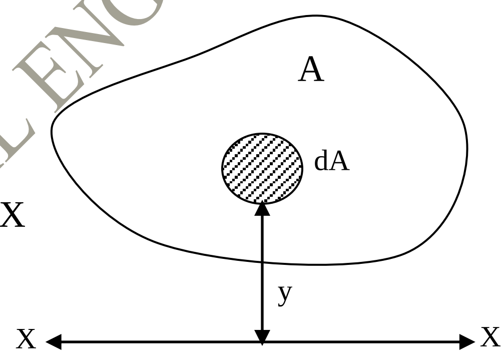
$$\bar{z} = \left(\frac{3R}{8}\right)$$

MOMENT OF INERTIA

Area Moment Of Inertia:-

Area moment of inertia is equal to second moment of area.

Consider an area situated in the space, where axis X-X is located as shown in figure.



Consider an elemental area dA which is located at y from X-X , then the Moment Of Inertia of dA about X-X axis is given by-

$$dI = dA.y^2$$

Then total Moment Of Inertia $I = \int dA.y^2$

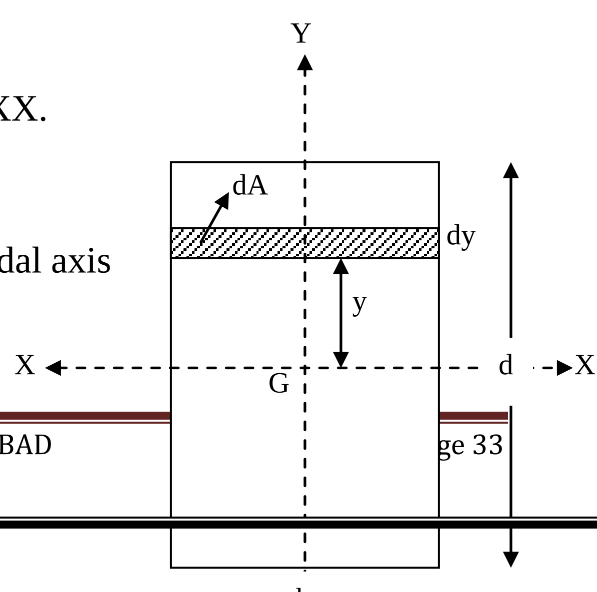
(1) Moment Of Inertia Of Rectangular Lamina About Centroidal Axis:

Consider a rectangle of width b and depth d.

Consider a strip of depth dy at a distance y from XX.

Area of the elemental strip dA = b.dy

Moment Of Inertia of the strip about the Centroidal axis XX is-



$$dI_{xx} = y^2.dA$$

Total Moment Of Inertia-

$$I_{xx} = \int_{-\frac{d}{2}}^{+\frac{d}{2}} y^2 dA = \int_{-\frac{d}{2}}^{+\frac{d}{2}} y^2.b.dy = \left[\frac{y^3}{3} b\right]_{-\frac{d}{2}}^{+\frac{d}{2}}$$

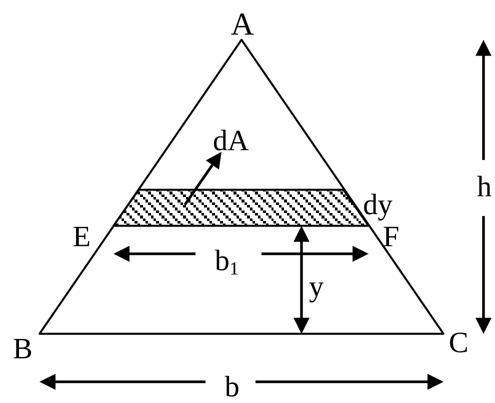
$$I_{xx} = b\left[\frac{d^3}{24} + \frac{d^3}{24}\right] = \left(\frac{bd^3}{12}\right)$$

$$\text{Similarly } I_{yy} = \left(\frac{db^3}{12}\right)$$

(2) <u>Moment Of Inertia Of A Triangle About Its Base:-</u>

Consider a triangle ABC of base b and height h.

Consider a strip of depth dy at a distance y from base.



From similar Δ AEF and ΔABC-

$$\frac{b_1}{b} = \frac{(h-y)}{h}$$

$$b_1 = \left(\frac{h-y}{h}\right) b = \left(1 - \frac{y}{h}\right) b$$

Area of strip = dA = b_1.dy = $\left(1 - \frac{y}{h}\right) b.dy$

Moment Of Inertia of this strip about base BC = $y^2 dA$

$$= y^2 \left(1 - \frac{y}{h}\right) b\, dy$$

Moment Of Inertia of the triangle about its base BC-

$$I_{BC} = \frac{b}{h}\int_0^h (hy^2 - y^3)\,dy = \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$I_{BC} = \frac{b}{h}\left[\frac{h^3}{3} - \frac{h^4}{4}\right]$$

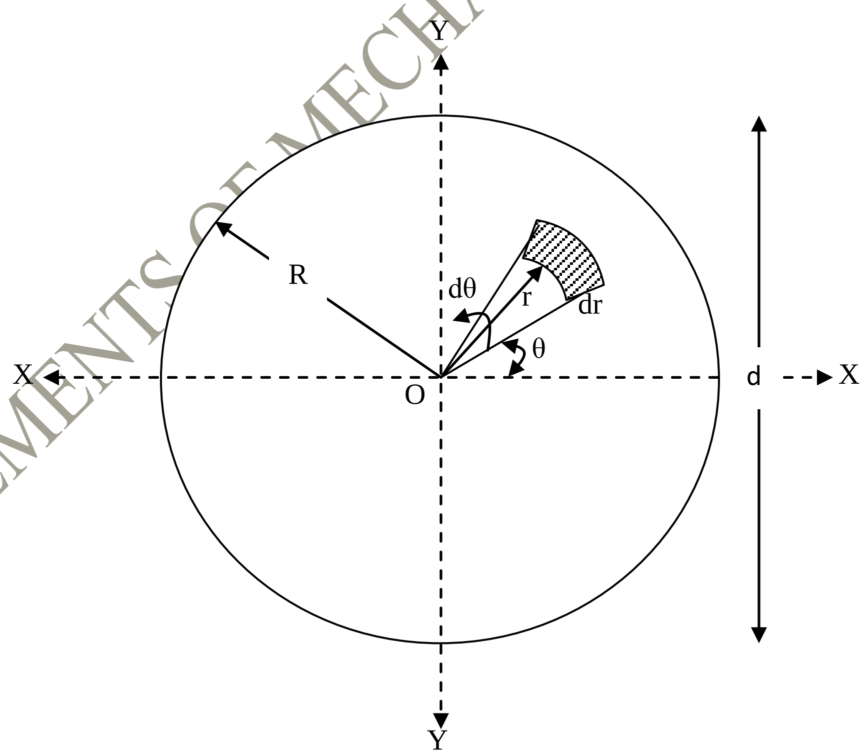
$$I_{BC} = \left(\frac{bh^3}{12}\right)$$

(3) Moment Of Inertia Of A Circle About Diametral Axis:-

METHOD-I

Consider a circle of radius R and centre O. Cut a element at radius r and thickness dr.

Area of element dA = rdθ.dr



Moment Of Inertia of element about diametral axis XX = y^2.dA

$$dI_{xx} = (r\sin\theta)^2 r d\theta . dr = r^3 \sin^2\theta \, d\theta \, dr$$

Moment Of Inertia of circle about diametral axis XX is given by-

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2\theta \, d\theta \, dr$$

$$I_{xx} = \left[\frac{r^4}{4}\right]_0^R \int_0^{2\pi} \left[\frac{(1 - \cos 2\theta)}{2}\right] d\theta$$

$$I_{xx} = \frac{R^4}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$I_{xx} = \left(\frac{\pi R^4}{4}\right) = \left(\frac{\pi d^4}{64}\right)$$

Similarly $I_{yy} = \left(\frac{\pi R^4}{4}\right) = \left(\frac{\pi d^4}{64}\right)$

And Moment Of Inertia about polar axis

$$I_{zz} = \left(\frac{\pi d^4}{32}\right)$$

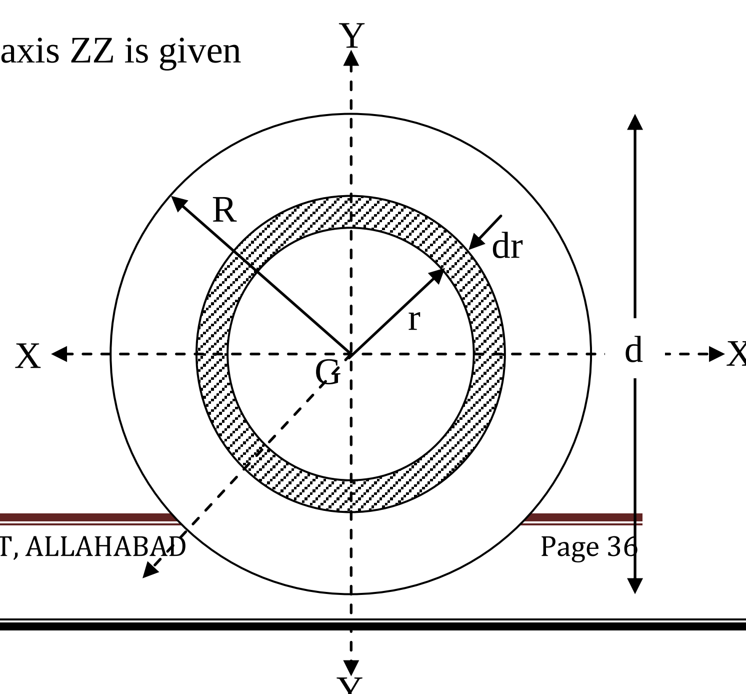
<u>METHOD-II</u>

Area of circular strip dA = 2πr.dr

Moment of inertia of dA about the polar axis ZZ is given by-

$dI_{zz} = dA . r^2 = (2\pi r \, dr) r^2 = 2\pi r^3 dr$

Moment of inertia of circle about the polar axis ZZ is-



$$I_{zz} = \int_0^R 2\pi r^3 dr = 2\pi \frac{R^4}{4} = \left(\frac{\pi d^4}{32}\right)$$

But $I_{xx} + I_{yy} = I_{zz}$

$$\therefore \quad I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \left(\frac{\pi d^4}{64}\right)$$

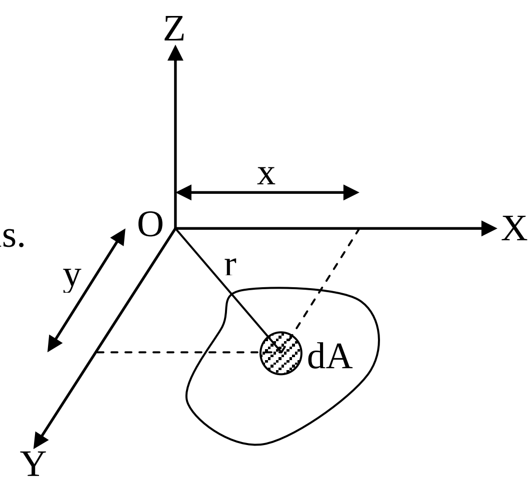
Perpendicular Axis theorem:-

Theorem of perpendicular axis states that's if I_{xx} and I_{yy} be the moment of inertia of a plane section about to mutually perpendicular axis X-X and Y-Y in the plane of section then the moment of inertia of the section I_{zz} about the axis Z-Z perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by-

$$I_{zz} = I_{xx} + I_{yy}$$

Moment of inertia I_{zz} is also known as Polar Moment Of Inertia

Proof:-



A plane section of area A and lying in plane X-Y as shown in figure.

Let OX and OY are two mutually perpendicular axis.

Consider a small area dA.

Let x = Distance of dA from the axis OY

y = Distance of dA from the axis OX

r = Distance of dA from the axis OZ

Then $r^2=x^2+y^2$

Now moment of inertia of dA about X-axis $dI_{xx} = dA.y^2$

Total moment of inertia about X-axis $I_{XX} = \sum dA.y^2$

Similarly-

Total moment of inertia about Y-axis $I_{YY} = \sum dA.x^2$

Total moment of inertia about Z-axis $I_{ZZ} = \sum dA.r^2$

$$I_{ZZ} = \sum dA.(x^2 + y^2)$$

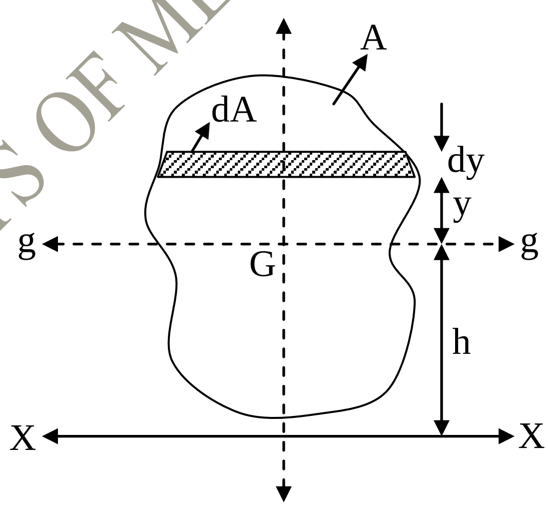
$$I_{ZZ} = \sum dA.y^2 + \sum dA.x^2$$

$$I_{ZZ} = I_{XX} + I_{YY}$$

Parallel Axis Theorem:-

This theorem states that's the moment of inertia of the area about non-centroidal axis is equal to the moment of inertia of the area about its centroidal axis which is parallel to the non-centroidal axis plus the product of area and square of the distance b/w the referred axis and axis passing through C.G. and parallel to the reference axis.

$$I_{xx} = I_{gg} + Ah^2$$



Proof:-

Moment of inertia of element about g-g axis

$$I_{gg} = \int dA.y^2$$

Moment of inertia of element about X-X axis

$$I_{xx} = \int dA(h + y)^2$$

$$I_{xx} = \int dA.y^2 + \int dA.h^2 + \int dA.2hy$$

$$I_{xx} = I_{gg} + Ah^2 + 2h\int dA.y$$

$$I_{xx} = I_{gg} + Ah^2 \qquad (\because \int dA.y = 0)$$

Note:- dA.y represent the moment of area of strip about g-g axis. $\int dA.y$ represent the moment of the total area about g-g axis. But moment of total area about g-g axis is equal to the product of the total area A and distance of C.G. of total area from g-g axis. As the distance of C.G. of the total area from g-g axis is zero hence $\int dA.y = 0$.