First-order logic

- First-order logic (FOL) models the world in terms of
  - **Objects,:**which are things with individual identities
  - **Properties :**of objects that distinguish them from other objects
  - **Relations :**that hold among sets of objects
  - **Functions:**which are a subset of relations where there is only one “value” for any given “input”
- Examples:
  - Objects: Students, lectures, companies, cars ...
  - Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
  - Properties: blue, oval, even, large, ...
  - Functions: father-of, best-friend, second-half, one-more-than ...

User provides

- **Constant symbols,** which represent individuals in the world
  - Mary
  - 3
  - Green
- **Function symbols,** which map individuals to individuals
  - father-of(Mary) = John
  - color-of(Sky) = Blue
- **Predicate symbols,** which map individuals to truth values
  - greater(5,3)
  - green(Grass)
  - color(Grass, Green)

FOL Provides

- **Variable symbols**
  - E.g., x, y, foo
- **Connectives**
  - Same as in PL: not ($\neg$), and ($\wedge$), or ($\vee$), implies ($\rightarrow$), if and only if (biconditional $\leftrightarrow$)
- **Quantifiers**
  - Universal $\forall$**x** or **(Ax)**
  - Existential $\exists$**x** or **(Ex)**

Sentences are built from terms and atoms

- A **term** (denoting a real-world individual) is a constant symbol, a variable symbol, or an n-place function of n terms.

  x and $f(x_1, ..., x_n)$ are terms, where each x_i is a term. A term with no variables is a **ground term**

- An **atomic sentence** (which has value true or false) is an n-place predicate of n terms

- A **complex sentence** is formed from atomic sentences connected by the logical connectives:

  $\neg P$, $P \vee Q$, $P \wedge Q$, $P \rightarrow Q$, $P \leftrightarrow Q$ where P and Q are sentences

- A **quantified sentence** adds quantifiers $\forall$ and $\exists$

- A **well-formed formula (wff)** is a sentence containing no "free" variables. That is, all variables are "bound" by universal or existential quantifiers.

  $(\forall x)P(x,y)$ has x bound as a universally quantified variable, but y is free.

Quantifiers

- **Universal quantification**
  - $(\forall x)P(x)$ means that P holds for **all** values of x in the domain associated with that variable
  - E.g., $(\forall x)$ dolphin(x) $\rightarrow$ mammal(x)
- **Existential quantification**
  - $(\exists\ x)P(x)$ means that P holds for **some** value of x in the domain associated with that variable
  - E.g., $(\exists\ x)$ mammal(x) $\wedge$ lays-eggs(x)
  - Permits one to make a statement about some object without naming it

Quantifier Scope

- Switching the order of universal quantifiers *does not* change the meaning:
  - $(\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x)\ P(x,y)$
- Similarly, you can switch the order of existential quantifiers:
  - $(\exists x)(\exists y)P(x,y) \leftrightarrow (\exists y)(\exists x)\ P(x,y)$
- Switching the order of universals and existentials *does* change meaning:
  - $(\forall x)(\exists y)$ likes(x,y)
  - $(\exists y)(\forall x)$ likes(x,y)

Connections between All and Exists

We can relate sentences involving $\forall$ and $\exists$ using De Morgan's laws:

$$(\forall x)\ \neg P(x) \leftrightarrow \neg(\exists x)\ P(x)$$

$$\neg(\forall x)\ P \leftrightarrow (\exists x)\ \neg P(x)$$

$$(\forall x)\ P(x) \leftrightarrow \neg\ (\exists x)\ \neg P(x)$$

$$(\exists x)\ P(x) \leftrightarrow \neg(\forall x)\ \neg P(x)$$

Translating English to FOL

Every gardener likes the sun.

You can fool some of the people all of the time. You can fool all of the people some of the time. All purple mushrooms are poisonous.

No purple mushroom is poisonous.

There are exactly two purple mushrooms.

Clinton is not tall.

X is above Y iff X is on directly on top of Y or there is a pile of one or more other objects directly on top of one another starting with X and ending with Y.

Translating English to FOL

Every gardener likes the sun.

$\forall x\ gardener(x) \rightarrow likes(x,Sun)$

You can fool some of the people all of the time.

$\exists x\ \forall t\ person(x) \wedge time(t) \rightarrow can\text{-}fool(x,t)$

You can fool all of the people some of the time.

$\forall x\ \exists t\ (person(x) \rightarrow time(t) \wedge can\text{-}fool(x,t))$ ← Equivalent → $\forall x\ (person(x) \rightarrow \exists t\ (time(t) \wedge can\text{-}fool(x,t))$

All purple mushrooms are poisonous.

$\forall x\ (mushroom(x) \wedge purple(x)) \rightarrow poisonous(x)$

No purple mushroom is poisonous.

$\neg\exists x\ purple(x) \wedge mushroom(x) \wedge poisonous(x)$

$\forall x\ (mushroom(x) \wedge purple(x)) \rightarrow \neg poisonous(x)$

(Equivalent)

There are exactly two purple mushrooms.

$\exists x\ \exists y\ mushroom(x) \wedge purple(x) \wedge mushroom(y) \wedge purple(y)\ \hat{}\ \neg(x=y) \wedge \forall z\ (mushroom(z) \wedge purple(z)) \rightarrow ((x=z) \vee (y=z))$

Clinton is not tall.

$\neg tall(Clinton)$

X is above Y iff X is on directly on top of Y or there is a pile of one or more other objects directly on top of one another starting with X and ending with Y.

$\forall x\ \forall y\ above(x,y) \leftrightarrow (on(x,y) \vee \exists z\ (on(x,z) \wedge above(z,y)))$

Example: A simple genealogy KB by FOL

- **Build a small genealogy knowledge base using FOL that**
  - contains facts of immediate family relations (spouses, parents, etc.)
  - contains definitions of more complex relations (ancestors, relatives)
  - is able to answer queries about relationships between people
- **Predicates:**
  - parent(x, y), child(x, y), father(x, y), daughter(x, y), etc.
  - spouse(x, y), husband(x, y), wife(x,y)
  - ancestor(x, y), descendant(x, y)
  - male(x), female(y)
  - relative(x, y)
- **Facts:**
  - husband(Joe, Mary), son(Fred, Joe)
  - spouse(John, Nancy), male(John), son(Mark, Nancy)
  - father(Jack, Nancy), daughter(Linda, Jack)
  - daughter(Liz, Linda)
  - etc.

- **Rules for genealogical relations**
  - $(\forall x,y)$ parent(x, y) $\leftrightarrow$ child (y, x)

    $(\forall x,y)$ father(x, y) $\leftrightarrow$ parent(x, y) $\wedge$ male(x) (similarly for mother(x, y))
    $(\forall x,y)$ daughter(x, y) $\leftrightarrow$ child(x, y) $\wedge$ female(x) (similarly for son(x, y))
  - $(\forall x,y)$ husband(x, y) $\leftrightarrow$ spouse(x, y) $\wedge$ male(x) (similarly for wife(x, y))
    $(\forall x,y)$ spouse(x, y) $\leftrightarrow$ spouse(y, x) (**spouse relation is symmetric**)
  - $(\forall x,y)$ parent(x, y) $\rightarrow$ ancestor(x, y)
    $(\forall x,y)(\exists z)$ parent(x, z) $\wedge$ ancestor(z, y) $\rightarrow$ ancestor(x, y)
  - $(\forall x,y)$ descendant(x, y) $\leftrightarrow$ ancestor(y, x)
  - $(\forall x,y)(\exists z)$ ancestor(z, x) $\wedge$ ancestor(z, y) $\rightarrow$ relative(x, y)
    (related by common ancestry)

    $(\forall x,y)$ spouse(x, y) $\rightarrow$ relative(x, y) (related by marriage)
    $(\forall x,y)(\exists z)$ relative(z, x) $\wedge$ relative(z, y) $\rightarrow$ relative(x, y)
    (**transitive**) $(\forall x,y)$ relative(x, y) $\leftrightarrow$ relative(y, x) **(symmetric)**
- **Queries**
  - ancestor(Jack, Fred) /* the answer is yes */
  - relative(Liz, Joe) /* the answer is yes */
  - relative(Nancy, Matthew)
    /* no answer in general, no if under closed world assumption */
  - $(\exists z)$ ancestor(z, Fred) $\wedge$ ancestor(z, Liz)

Semantics of FOL

- **Domain M**: the set of all objects in the world (of interest)
- **Interpretation I**: includes
  - Assign each constant to an object in M
  - Define each function of n arguments as a mapping M^n => M
  - Define each predicate of n arguments as a mapping M^n => {T, F}
  - Therefore, every ground predicate with any instantiation will have a truth value
  - In general there is an infinite number of interpretations because |M| is infinite
- **Define logical connectives: ~, ^, v, =>, <=>** as in PL
- **Define semantics of (∀x) and (∃x)**
  - (∀x) P(x) is true iff P(x) is true under all interpretations
  - (∃x) P(x) is true iff P(x) is true under some interpretation

- **Model**: an interpretation of a set of sentences such that every sentence is *True*
- **A sentence is**
  - **satisfiable** if it is true under some interpretation
  - **valid** if it is true under all possible interpretations
  - **inconsistent** if there does not exist any interpretation under which the sentence is true
- **Logical consequence**: S |= X if all models of S are also models of X

Resolution

- Reminder: Resolution rule for propositional logic:
  - $P_1 \vee P_2 \vee ... \vee P_n$
  - _ $\neg P_1 \vee Q_2 \vee ... \vee Q_m$
  - Resolvent: $P_2 \vee ... \vee P_n \vee Q_2 \vee ... \vee Q_m$
- Examples
  - P and $\neg$ P $\vee$ Q : derive Q (Modus Ponens)
  - ($\neg$ P $\vee$ Q) and ($\neg$ Q $\vee$ R) : derive $\neg$ P $\vee$ R
  - P and $\neg$ P : derive False [contradiction!]
  - (P $\vee$ Q) and ($\neg$ P $\vee$ $\neg$ Q) : derive True

Resolution in first-order logic

- Resolution is **sound** and **complete** for FOL
- Given sentences

  $P_1 \vee \ldots \vee P_n$ and $Q_1 \vee \ldots \vee Q_m$
- in *conjunctive normal form:*
  - each P_i and Q_i is a literal, i.e., a positive or negated predicate symbol with its terms,
- if P_j and $\neg Q_k$ **unify** with substitution list θ, then derive the resolvent sentence:

  $\text{subst}(\theta, P_1 \vee \ldots \vee P_{j-1} \vee P_{j+1} \ldots P_n \vee Q_1 \vee \ldots Q_{k-1} \vee Q_{k+1} \vee \ldots \vee Q_m)$
- Example
  - from clause $\mathbf{P(x, f(a)) \vee P(x, f(y)) \vee Q(y)}$
  - and clause $\mathbf{\neg P(z, f(a)) \vee \neg Q(z)}$
  - derive resolvent $\mathbf{P(z, f(y)) \vee Q(y) \vee \neg Q(z)}$
  - using $\boldsymbol{\theta = \{x/z\}}$

Resolution refutation

- Given a consistent set of axioms KB and goal sentence Q, show that KB |= Q
- **Proof by contradiction:** Add ¬Q to KB and try to prove false.

  i.e., (KB |- Q) ↔ (KB ∧ ¬Q |- False)
- Resolution is **refutation complete:** it can establish that a given sentence Q is entailed by KB, but can't (in general) be used to generate all logical consequences of a set of sentences

Resolution example

- KB:
  - allergies(X) $\rightarrow$ sneeze(X)
  - cat(Y) $\wedge$ allergic-to-cats(X) $\rightarrow$ allergies(X)
  - cat(Felix)
  - allergic-to-cats(Lise)
- Goal:
  - sneeze(Lise)

Refutation resolution proof tree