Optimum Receivers for the AWGN channel

Contents

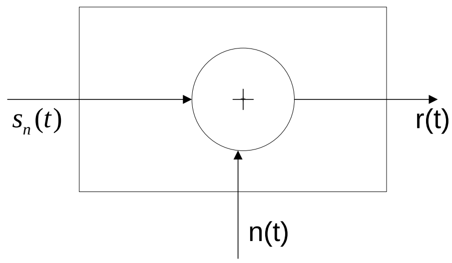
AWGN: Additive White Gaussian Noise

Objective :

- Receiver design
- Performance evaluation (memory, No memory)

Note: These notes are preliminary and are posted by the request of the students. Please, report to me all the mistakes that you find in the document. (muqaibel@kfupm.edu.sa). This material is for the sole purpose of in class usage. Please observe the copyright of the original authors for any content in these notes.

Optimum Receiver

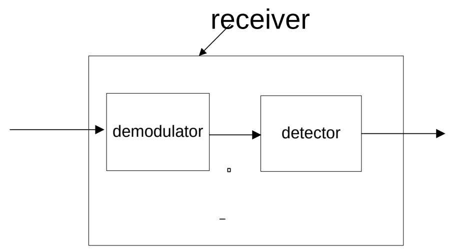


$$r(t) = s_m(t) + n(t) \qquad 0 \le t \le T$$

$$\{s_m(t), m = 1, 2, ..., M\}$$

$$S_{nm}(f) = \frac{1}{2} N_0 \qquad W / Hz$$

Objective: Upon observation of *r(t)*, what is the optimum receiver "design" in terms of probability of making error.



Demodulator : converts the received waveform r(t) into an N-dimensional vector $\bar{r} = [r_1 \quad r_2 \quad \dots \quad r_N]$ where *N* is the dimension of the transmitted signal.

Detector: Is to decide which of the M possible signals waveforms was transmitted based on $\bar{r}$.

Types of detectors

1. The optimum detector
2. Maximum likelihood sequence detector
3. Symbol by symbol MAP

Optimum Demodulator

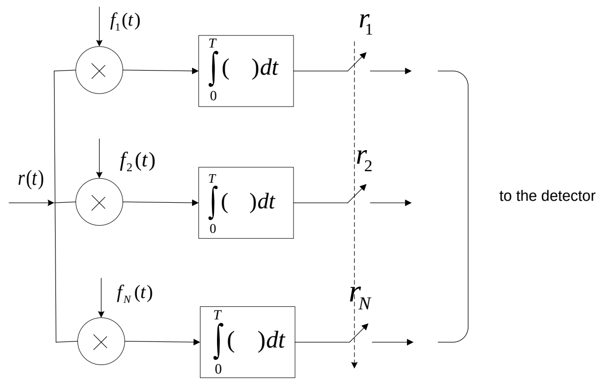
1. Correlators
2. Matched filters

Correlation Demodulator

Decomposes the received signal-to-noise ratio into N-dimensional vectors.

- Linearly weighted orthogonal basis functions $\{f_n(t)\}$, where $\{f_n(t)\}$ spans the signal space but not the noise space.

We can show that the noise terms that falls outside the signal space is irrelevant to the detection.



$$\int_0^T r(t) f_k(t)dt = \int_0^T [s_m(t) + n(t)] f_k(t)dt$$

$$r_k = s_{mk} + n_k \qquad k = 1, 2, \ldots N$$

where

$$s_{mk} = \int_0^T s_m(t) f_k(t)dt, \qquad k = 1, 2, \ldots N$$

$$n_k = \int_0^T n(t) f_k(t)dt, \qquad k = 1, 2, \ldots N$$

$$r(t) = \sum_{k=1}^{N} s_{mk} f_k(t) + \sum_{k=1}^{N} n_k f_k(t) + n'(t) = \sum_{k=1}^{N} r_k f_k(t) + n'(t)$$

$$\Rightarrow n'(t) = n(t) - \sum_{k=1}^{N} n_k f_k(t)$$

$n'(t)$ is zero mean Gaussian noise process. It is the unrepresented part of noise "inherent".
$n'(t)$ is Gaussian because it is the sampled output of a linear filter excited by a Gaussian input.

$$E(n_n) = \int_0^T E[n(t)] f_k(t)dt = 0 \qquad for\ all\ k$$

$$E(n_k m_k) = \int_0^T \int_0^T E[n(\tau)n(\tau)] f_k(t) f_m(\tau)dtd\tau$$

$$= \frac{1}{2} N_0 \int_0^T \int_0^T \delta(t-\tau) f_k(t) f_m(\tau)dtd\tau$$

$$= \frac{1}{2} N_0 \int_0^T f_k(t) f_m(\tau)dt = \frac{1}{2} N_0 \delta_{mk}$$

$$\delta_{mk} = 1$$

$\delta_{mk} = 1$, when m=k and zero otherwise.

- $\{n_k\}$ are zero mean uncorrelated random variable with common covariance

$$\sigma_n^2 = \frac{1}{2}N_0$$

$$E[r_k] = E[s_{mk} + n_k] = s_{nk}$$

$$\sigma_r^2 = \sigma_n^2 = \frac{1}{2}N_0$$

Gaussian uncorrelated implies statistically independent.

$$\bar{r} = [r_1 \quad r_2 \quad \ldots \quad r_N]$$

$$p(\bar{r}/\bar{s}_m) = \prod_{k=1}^{N} p(r/s_{mk}) \qquad m = 1, 2, ..., M$$

$$when \qquad p(r/s_{mk}) = \frac{1}{\sqrt{\pi N_0}} \exp[\frac{-(r_k - s_{mk})^2}{N_0}] \qquad , k = 1, 2, ...N$$

The joint conditional pdf

$$p(\bar{r}/\bar{s}_m) = \frac{1}{(\pi N_0)^{N/2}} \exp[-\sum_{k=1}^{N} \frac{(r_k - s_{mk})^2}{N_0}], \qquad m = 1, 2, ..., M$$

As a final step we can show that $(r_1, r_2, ..., r_N)$ are sufficient statistics.

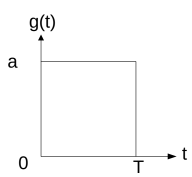
No additional relevant information can be extracted from $n'(t)$.

$E[n'(t) r_k] = 0$ uncorrelated proof p 235

Gaussian and uncorrelated implies statistically independent which implies ignore $n'(t)$

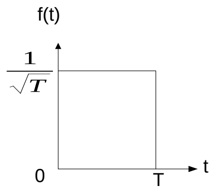
Example

M-ary PAM



$$E_g = \int_0^T g^2(t)dt = \int_0^T a^2 dt = a^2 T$$

PAM one basis function



$$f(t) = \frac{1}{\sqrt{a^2 T}} g(t) = \begin{cases} 1/\sqrt{T} & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

The output of the correlator

$$r = \int_0^T r(t) f(t) dt = \frac{1}{\sqrt{T}} \int_0^T r(t) dt$$

The correlator becomes a simple integrator when f(t) is rectangular .

$$r = \frac{1}{\sqrt{T}} \{ \int_0^T [s_m(t) + n(t)\} dt = \frac{1}{\sqrt{T}} [\int_0^T s_m(t) dt + \int_0^T n(t) dt]$$

$$r = s_m + n$$

$$E[n] = 0$$

$$\sigma_n^2 = E[\frac{1}{T} \int_0^T \int_0^T n(t) n(\tau) dt d\tau] = \frac{1}{T} \int_0^T \int_0^T E[n(t) n(\tau)] dt d\tau$$

$$= \frac{N_0}{2T} \int_0^T \int_0^T \delta(t - \tau) dt d\tau = \frac{1}{2} N_0$$

pdf of the sampled output $\quad p(r/s_m) = \frac{1}{\sqrt{\pi N_0}} \exp[\frac{-(r - s_m)^2}{N_0}]$

Matched filter Demodulator

We use N filters

$$h_k(t) = \begin{cases} f_k(T - t) & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

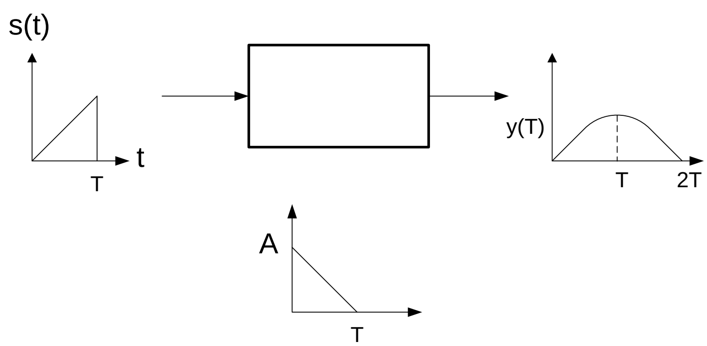
$$y_k(t) = \int_0^t r(\tau) h_k(t - \tau) d\tau$$

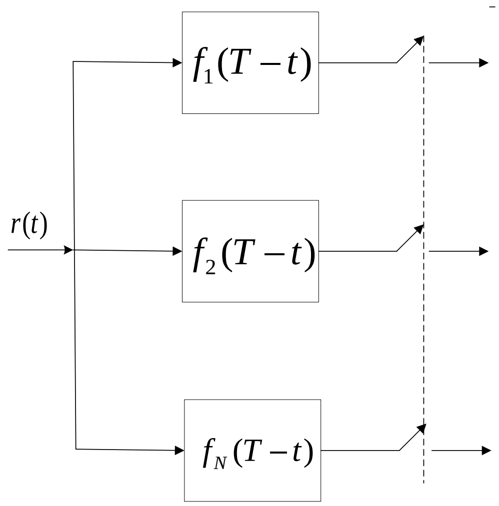
$$= \int_0^t r(\tau) f_k(T - t + \tau) d\tau \qquad k = 1, 2, ...N$$

If we sample at the end the period t=T

$$y_k(T) = \int_0^T r(\tau) f_k(\tau) d\tau = r_k \qquad k = 1, 2, \ldots N$$

Matched filter: A filter whose impulse response h(t)=s(T-t) or s(t) where s(t) is assumed to be confined to the time interval 0≤t≤T.





Property of Matched filter:_ If a signal s(t) is corrupted by AWGN, the filter with an impulse response matched to s(t) maximizes the output signal to noise ratio (SNR).
Proof :

$$y(t)=\int_0^t r(\tau)h(t-\tau)d\tau=\int_0^t s(\tau)h(t-\tau)d\tau+\int_0^t n(\tau)h(t-\tau)d\tau \qquad at \quad t=T$$

$$y(T)=\int_0^T s(\tau)h(T-\tau)d\tau+\int_0^T n(\tau)h(T-\tau)d\tau$$

$$=y_s(T)+y_n(T)$$

$$SNR_0=\frac{y_s^2(T)}{E[y_n^2(T)]}$$

$$where \qquad E[y_n^2(T)] \qquad noise\ variance$$

$$E[y_n^2(T)]=\int_0^T\int_0^T E[n(\tau)n(t)]h(T-\tau)h(T-t)dtd\tau$$

$$=\frac{1}{2}N_0\int_0^T\int_0^T \delta(t-\tau)h(T-\tau)h(T-t)dtd\tau$$

$$=\frac{1}{2}N_0\int_0^T h^2(T-t)dt$$

$$SNR_0=\frac{[\int_0^T s(\tau)h(T-\tau)d\tau]^2}{\frac{1}{2}N_0\int_0^T h^2(T-t)dt}=\frac{[\int_0^T h(\tau)s(T-\tau)d\tau]^2}{\frac{1}{2}N_0\int_0^T h^2(T-t)dt}$$

Can we maximize the numerator while the denominator is held constant.

Cauchy-Schwarz inequality:

$$[\int_{-\infty}^{\infty} g_1(t)g_2(t)dt]^2\le\int_{-\infty}^{\infty} g_1{}^2(t)dt\int_{-\infty}^{\infty} g_2{}^2(t)dt$$

With equality when $g_1(t)=c\,g_2(t)$

$g_1(t)=h(t)$, $g_2(t)=s(T-t)$ more when $h(t)=c\,s(T-t)$, c^2 drop from the numerator and denominator

$$SNR_0=\frac{2}{N_0}\int_0^T s^2(t)dt=\frac{2E}{N_0}$$

Property: The output SNR from the matched filter depends on the energy of the waveform s(t) but not on the details (shape) of s(t).

Frequency domain interpretation of matched filter

$$Y(f) = |S(f)|^2 e^{-j2\pi ft}$$

$$Y_s(t) = \int_{-\infty}^{\infty} Y(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} |S(f)|^2 e^{-j2\pi ft} e^{j2\pi ft} df$$

$$at\, t = T$$

$$Y_s(T) = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_{-\infty}^{\infty} s^2(t)dt = E \qquad Pasaval's\, relation$$

Noise at the output of the matched filter

$$\Phi(f) = \frac{1}{2}|H(f)|^2 N_0$$

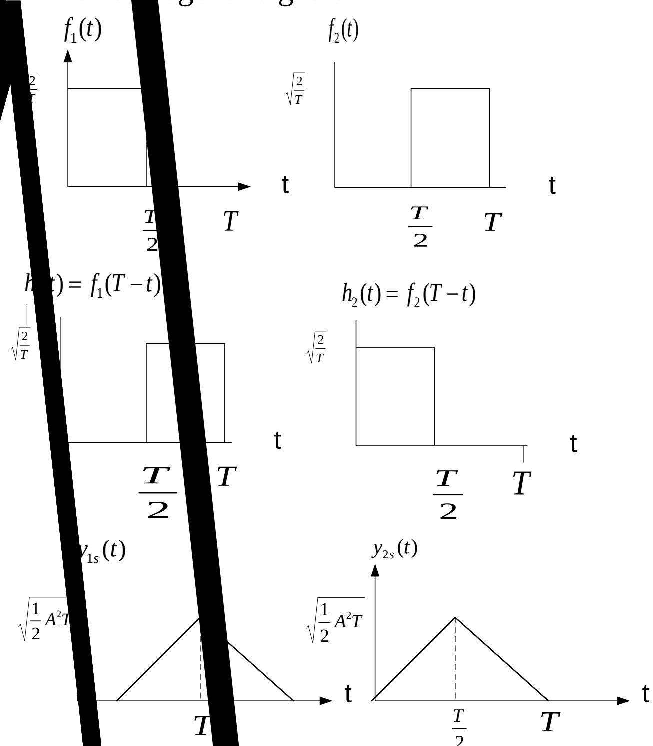
$$P_n = \int_{-\infty}^{\infty} \Phi(f)df = \frac{1}{2} N_0 \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{1}{2} N_0 \int_{-\infty}^{\infty} |S(f)|^2 df = \frac{1}{2} E N_0$$

$$P_s = Y_s^2(T) \Rightarrow SNR_0 = \frac{P_s}{P_n} = \frac{E^2}{\frac{1}{2} E N_0} = \frac{2E}{N_0}$$

Matched filter and correlator at equivalent at t=T but matched filter is immune to time jitters.

Example 1.2

M=4 bi-orthogonal signals



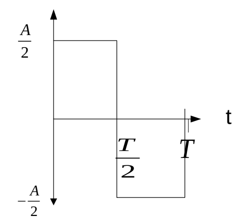
Note the response to $s_1(t)$ is evaluated at T

$$\bar{r} = [r_1 \quad r_2] = [\sqrt{E} + n_1 \quad n_2]$$

$$SNR = \frac{(\sqrt{E})^2}{\frac{1}{2}N_0} = \frac{2E}{N_0}$$

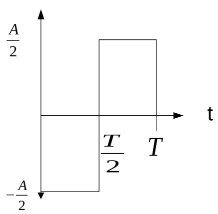
Additional Example (Matched Filters)

Consider the signal $s(t)$



a) Determine the impulse response of a filtered matched to this signal and sketch it.

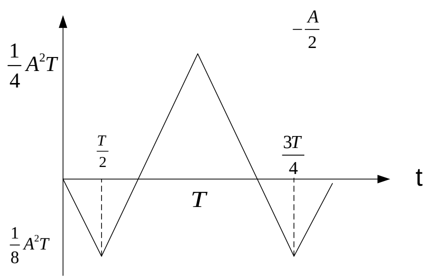
$h(t)=s(T-t)$



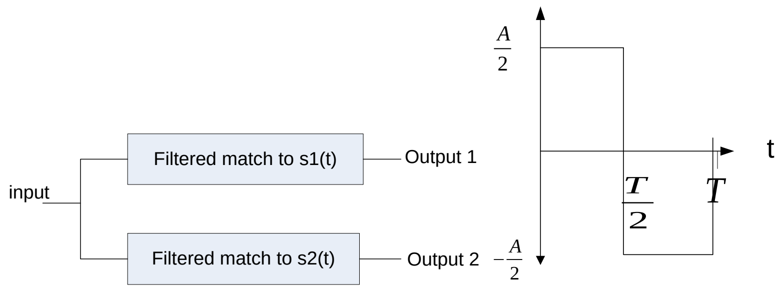
b) Determine the matched filter output as function of time.

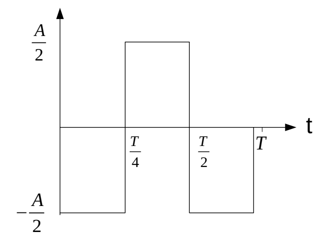
c) This is done by convolution, we may ask about the values at the peak (figure)

$A^2T/4 = E^2$



A pair of pulses that are orthogonal to each other over the interval [0,T], are used for two dimensional matched filter





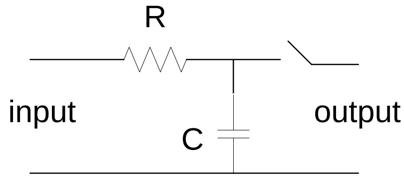
What is the output of the matched filter at T?

a) When $s_1(t)$ is applied to the two branches.
b) When $s_2(t)$ is applied to the two branches.

First branch is shown in problem 4.1
Second branch is zero [generalize to all orthogonal signals].

Additional Problems on Matched filters.

Another method for approximating "realization" of matched filter is the (RC) low pass filter [integrator].

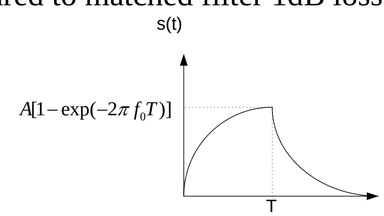


$$H(f)=\frac{1}{1+j\, f/f_0},\quad f_0=\frac{1}{2\pi RC}$$

The input signal is rectangular of pulse amplitude A and duration T.

Objective: Optimize the selection of 3-dB cutoff frequency f_0 of the filter. So that the peak output SNR is maximized.

Show that $f_0=0.2/T$ is the optimum. Compared to matched filter 1dB loss.



The peak value of the output power is

$$P_{out}=A^2[1-\exp(-2\pi f_0 T)]^2$$

f_0 is the 3-dB cutoff frequency of the RC filter.

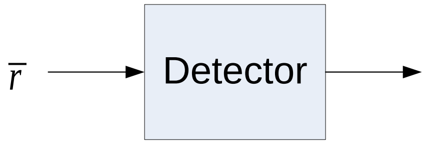
$$N_{out}=\frac{N_0}{2}\int_{-\infty}^{\infty}\frac{df}{1+(f/f_0)^2}=\frac{N_0\pi f_0}{2}$$

The corresponding value for the SNR

$$(SNR)_{out}=\frac{2A^2}{N_0\pi f_0}[1-\exp(-2\pi f_0 T)]$$

Differentiating with respect to (f_0T) and setting the result equal to zero.
The maximum value of $(SNR)_{out}$ is at $f_0=0.2/T$.

The Optimum Detector



assume no memory

Objective: Maximize the probability of correct decisions.
Posterior probabilities.
P(signal $\bar{s}_m$ was transmitted| $\bar{r}$) = $P(\bar{s}_m \mid \bar{r})$ where m =1,2,..M
Hence the name maximum a posteriori probability (MAP)

Using Baye's rule: $P(\bar{s}_m \mid \bar{r}) = \dfrac{P(\bar{r} \mid \bar{s}_m) P(\bar{s}_m)}{P(\bar{r})}$

$P(\bar{s}_m)$ a priori probability of the m^{th} signal.

$$P(\bar{r}) = \sum_{m=1}^{M} P(\bar{r} \mid \bar{s}_m) P(\bar{s}_m)$$

When the M-signals are equally probable $P(\bar{s}_m)$ =1/M for all m.

The same rule that maximizes $P(\bar{s}_m \mid \bar{r})$ is equivalent to maximizing $P(\bar{r} \mid \bar{s}_m)$

Likelihood function: is the conditional PDF $P(\bar{r} \mid \bar{s}_m)$ or any monotonic function of it.

⇨ Maximum –likelihood (ML) criterion.

MAP = ML if { $\bar{s}_m$ } is equi-probable.

For AWGN , the likelihood function is given by 5.1.12

$$P(\bar{r} \mid \bar{s}_m) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^{N} \frac{(r_k - s_{mk})^2}{N_0} \right]$$

or

$$\ln P(\bar{r} \mid \bar{s}_m) = -\frac{1}{2} N - \ln(\pi N_0) - \frac{1}{N_0} \sum_{k=1}^{N} (r_k - s_{mk})^2$$

Maximizing $\ln P(\bar{r} \mid \bar{s}_m)$ is equivalent to minimizing

$$D(\bar{r} \mid \bar{s}_m) = \sum_{k=1}^{N} (r_k - s_{mk})^2$$

where

$D(\bar{r} \mid \bar{s}_m)$ is Euclidean distance and m= 1,2,…M

Minimum Distance Detection

Another interpretation of ML criterion.

$$D(\bar{r},\bar{s}_m)=\sum_{n=1}^{N} r_n^{\,2}-2\sum_{n=1}^{N} r_n s_{mn}+\sum_{n=1}^{N} s_{mn}^{\,2}$$

$$=\left\|\bar{r}\right\|^2-2\,\bar{r}.\bar{s}_m+\left\|\bar{s}_m\right\|^2 \quad \text{where m = 1,2,3…M}$$

$$D'(\bar{r},\bar{s}_m)=-2\,\bar{r}.\bar{s}_m+\left\|\bar{s}\right\|^2$$

Let $C(\bar{r},\bar{s}_m)=-D'(r,s_m)$

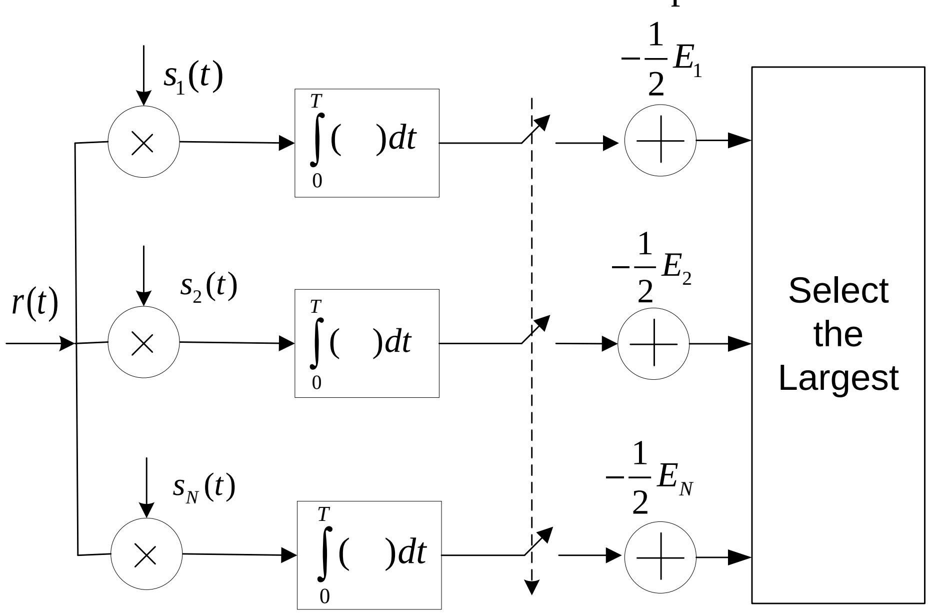
To get rid of the minus. Now, we maximize C rather than minimize D.

$$C(\bar{r},\bar{s}_m)=2.\bar{r}.\bar{s}_m-\left\|\bar{s}_m\right\|^2$$

$\left\|\bar{s}_m\right\|^2$ can be eliminated if energy is fixed for all m =1,2,…M, but cannot if signals have uequal energy (PAM).

$$C(\bar{r},\bar{s}_m)=2\int_0^T r(t)\,s_m(t)dt-E_m \quad \text{m= 0,1,2,…,M}$$

An alternative realization of the optimum AWGN receiver.



Summary:

Optimum ML

1) compute $D(\bar{r},\bar{s}_m)$ or $D'(\bar{r},\bar{s}_m)$ → distance metrics and chooses the smallest or

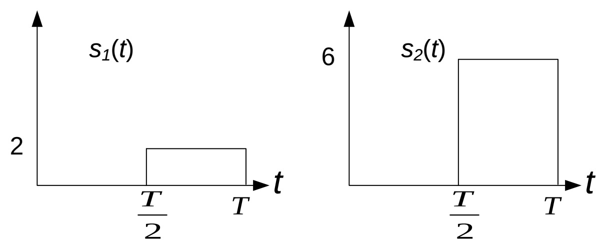
2) compute $C(\bar{r},\bar{s}_m)$ → correlator metrics and choose the largest

3) ML= MAP if eqiprobable { $\bar{s}_m$ }, otherwise

$$PM(\bar{r},\bar{s}_m) = P(\bar{r}\,|\,\bar{s}_m)P(\bar{s}_m)$$

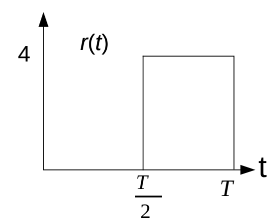
Example:

Let us assume that we are sending one of two levels either 2 or 6 as shown in the figure. To illustrate the importance of subtracting the energy of the symbol. We will consider two cases. The first one will assume that the received amplitude is 4 and then we will consider the amplitude to be 3.



$$C(\bar{r},\bar{s}_m) = 2.\bar{r}.\bar{s}_m - \left\|\bar{s}_m\right\|^2$$

For the middle case with amplitude 4 we get a fair comparison after removing the bias

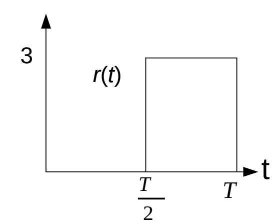


$$C(\bar{r},\bar{s}_m) = 2.\bar{r}.\bar{s}_m - \left\|\bar{s}_m\right\|^2$$

$$C(\bar{r},\bar{s}_1) = 2.(4).(2) - 4 = 16 - 4 = 12$$

$$C(\bar{r},\bar{s}_2) = 2.(4).(6) - 36 = 48 - 36 = 12$$

For the case that the received signal with amplitude of 3 we get the wrong decision unless we remove the bias



$$C(\bar{r},\bar{s}_1) = 2.(3).(2) - 4 = 12 - 4 = 8$$

$$C(\bar{r},\bar{s}_2) = 2.(3).(6) - 36 = 36 - 36 = 0$$

Example:

In binary PAM $s_1 = -s_2 = \sqrt{E_b}$

Prior probabilities $P(s_1) = \rho$, $P(s_2) = 1-\rho$

Determine the metrics of the optimum MAP for AWGN.

$r = \pm\sqrt{E_b} + y_n(T)$ ← zero mean and $\sigma_n^2 = \frac{1}{2}N_0$

Note: the variance of the sampled noise is $N_0/2$. In general the noise power, $P=N_0B$, According to Nyquist $B=1/(2T)$, When looking at the energy we $E=PT$

$$P(r\,|\,s_1) = \frac{1}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{(r-\sqrt{E_b})^2}{2\sigma_n^2}\right]$$

$$P(r\,|\,s_2) = \frac{1}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{(r+\sqrt{E_b})^2}{2\sigma_n^2}\right]$$

$$PM(\bar{r},\bar{s}_1) = \rho\, p(r\,|\,s_1) = \frac{\rho}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{\left(r-\sqrt{E_b}\right)^2}{2\sigma_n^2}\right]$$

$$PM(\bar{r},\bar{s}_2) = (1-\rho)\, p(r\,|\,s_2) = \frac{1-\rho}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{\left(r+\sqrt{E_b}\right)^2}{2\sigma_n^2}\right]$$

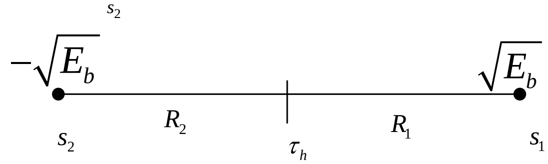
If $PM(\bar{r},\bar{s}_1) > PM(\bar{r},\bar{s}_2)$ then choose s_1

$$\frac{PM(\bar{r},\bar{s}_1)}{PM(\bar{r},\bar{s}_2)} \underset{s_2}{\overset{s_1}{\gtrless}} 1$$

$$\Rightarrow \frac{PM(\bar{r},\bar{s}_1)}{PM(\bar{r},\bar{s}_2)} = \frac{\rho}{1-\rho}\exp\left[\frac{(r+\sqrt{E_b})^2-(r-\sqrt{E_b})^2}{2\sigma_n^2}\right] \underset{s_2}{\overset{s_1}{\gtrless}} 1$$

$$\left[\frac{(r+\sqrt{E_b})^2-(r-\sqrt{E_b})^2}{2\sigma_n^2}\right] \underset{s_2}{\overset{s_1}{\gtrless}} \ln\frac{\rho}{1-\rho}$$

$$\sqrt{E_b}\,r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{1}{2}\sigma_n^2\ln\frac{\rho}{1-\rho} = \frac{1}{4}N_0\ln\frac{\rho}{1-\rho}$$



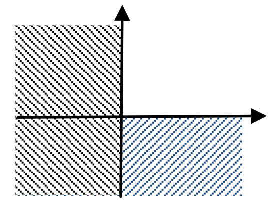
1) If $\rho = ½$, $\tau_h = 0$
2) If $\rho \neq ½$, knowledge of N_0 or $\frac{N_0}{E_b}$ is required for optimal detection.

If the M signals are equi-probable.

The maximum Likelihood (ML) minimize P(correct decision)

$$P(c)=\sum_{m=1}^{M}\frac{1}{M}\int_{R_m} p(\bar{r}\,|\,\bar{s}_m)d\bar{r} \qquad ML$$

$$P(c)=\sum_{m=1}^{M}\frac{1}{M}\int_{R_m} p(\bar{s}_m\,|\,\bar{r})\,p(\bar{r})d\bar{r} \qquad MAP$$



R_m is the region for correct decision. Explain the Union bound Concept of QPSK

P(e)=1-P(c)

The Maximum Likelihood Sequence Detector

If no memory *(symbol-by-symbol detector)* is optimal (minimum probability of error)
Memory => successive symbols are interdependent.

The *maximum likelihood (ML) sequence detector*: searches for the minimum Euclidean distance path through the trellis that characterizes the memory in the transmitted sequence.

Example NRZI

(PAM) (zero : like before, one : flip)

$s_1 = -s_2 = \sqrt{E_b}$

$r_k = \pm\sqrt{E_b} + n_k$

$$P(r_k\,|\,s_1)=\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{(r_k-\sqrt{E_b})^2}{2\sigma_n^2}\right]$$

$$P(r_k\,|\,s_2)=\frac{1}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{(r_k+\sqrt{E_b})^2}{2\sigma_n^2}\right]$$

$$P(r_1,r_2,\ldots r_k\,|\,s^{(m)})=\prod_{k=1}^{K}P(r_k\,|\,s_k^{(m)})=\left(\frac{1}{\sqrt{2\pi}\sigma_n}\right)^k\exp\left[-\sum_{k=1}^{K}\frac{(r_k-s_k^{(m)})^2}{2\sigma_n^2}\right]$$

Maximize the above probability.
By taking the logarithm and consider only those relevant term.

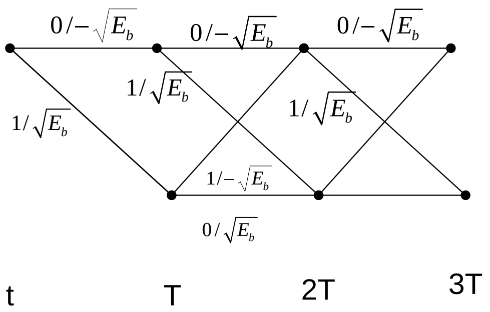
$$D(\bar{r},\bar{s}^{(m)})=\sum_{k=1}^{K}(r_k-s_k^{(m)})^2$$

For binary we need to search 2^k sequences, where k is the sequence length.

Viterbi algorithm: is a sequential trellis search algorithm for performing ML sequence detection

- ⇨ Used for decoding "convolutional codes"
- ⇨ Assume initial state s_0

The example of Binary NRZI



At $2T$ and so on there are two arrows entering the state, we choose the minimum distance "*survivor*".

$$D_0(0,0) = (r_1 + \sqrt{E_b})^2 + (r_2 + \sqrt{E_b})^2$$

$$D_0(1,1) = (r_1 - \sqrt{E_b})^2 + (r_2 + \sqrt{E_b})^2$$

For state '1' we do similar

$$D_1(0,1) = (r_1 + \sqrt{E_b})^2 + (r_2 - \sqrt{E_b})^2$$

$$D_1(1,0) = (r_1 - \sqrt{E_b})^2 + (r_2 - \sqrt{E_b})^2$$

At t=3T Suppose the survivors are (0,0) and (0,1)

$$D_0(0,0,0) = D_0(0,0) + (r_3 + \sqrt{E_b})^2$$

$$D_0(0,1,1) = D_1(0,1) + (r_3 + \sqrt{E_b})^2$$

and

$$D_1(0,0,1) = D_0(0,0) + (r_3 - \sqrt{E_b})^2$$

$$D_1(0,1,0) = D_1(0,1) + (r_3 - \sqrt{E_b})^2$$

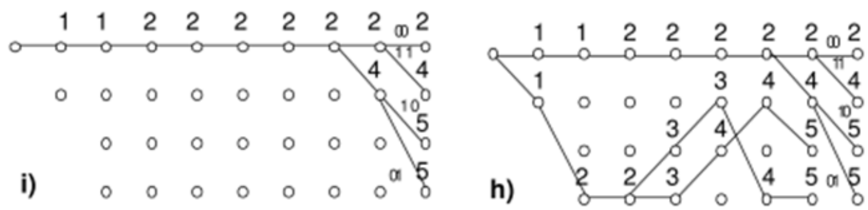
Using Viterbi in this example the number of path searched is reduced by a factor of two at each stage.

The memory length is L (L=1 in the previous example)

You make a decision when the survivor path agree. "Variable delay" negative ??

Practically at 5L and then make decision at k even then before k-5L is almost identical.

⇨ Make a decision.



Example (in the previous NRZI)

Let E_b=1 and the demodulated sequence is (0.9,-0.8,0.3,-1.1,1.2,1.5,-0.7)

$$D_0(0,0) = (0.9+1)^2 + (-0.8+1)^2 = 1.9^2 + 0.2^2 = 3.61 + 0.04 = 3.65$$

$$D_0(1,1) = (0.9-1)^2 + (-0.8+1)^2 = (-0.1)^2 + 0.2^2 = 0.01 + 0.04 = 0.05$$

$$D_1(0,1)=(0.9+1)^2+(-0.8-1)2=1.9^2+1.8^2=3.61+3.24=6.85$$

$$D_1(1,0)=(0.9-1)^2+(-0.8-1)2=0.1^2+1.8^2=0.01+3.24=3.25$$

$$D_0(1,1,0) = D_0(1,1) + (0.3+1)^2 = 0.05 + 1.69 = 1.74$$

$$D_0(1,0,1) = D_0(1,0) + (0.3-1)^2 = 3.25 + 0.49 = 3.74$$

$$D_1(1,1,1) = D(1,1) + (0.3-1)^2 = 0.05 + 0.49 = 0.54$$

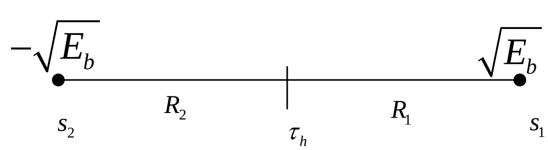
$$D_1(1,0,0) = D(1,0) + (0.3+1)^2 = 3.25 + 1.69 = 4.94$$

A symbol by symbol MAP detector for signals with memory.

- Optimum (minimize symbol error)
- If M is large => large computational complexity.
- Used for convolution codes and turbo coding.
- Beyond the scope.

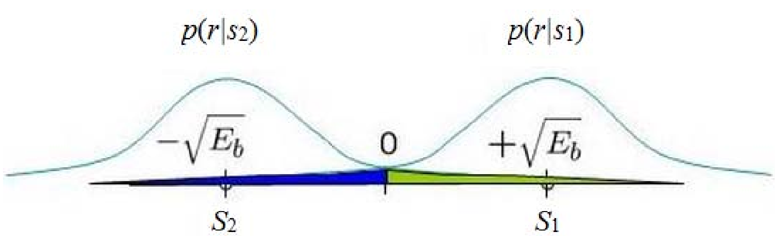
Performance of the Optimum Receiver for Memory less Modulation

Probability of Error for Antipodal Binary Modulation



PAM antipodal $s_1(t)= -s_2(t)$

$r = s_1 + n = \sqrt{E_b} + n$ AWGN



$$P(r\,|\,s_1)=\frac{1}{\sqrt{\pi N_0}}e^{-(r-\sqrt{E_b})^2/N_0}$$

$$P(r\,|\,s_2)=\frac{1}{\sqrt{\pi N_0}}e^{-(r+\sqrt{E_b})^2/N_0}$$

$$P(e\,|\,s_1)=\int_{-\infty}^{0}P(r\,|\,s_1)dr=\frac{1}{\sqrt{\pi N_0}}\int_{-\infty}^{0}e^{-(r+\sqrt{E_b})^2/N_0}dr$$

By replacing variables $x=\frac{r-\sqrt{E_b}}{\sqrt{N_0/2}}$, $dx=\sqrt{\frac{2}{N_0}}dr \Rightarrow dr=\sqrt{\frac{N_0}{2}}dx$

$$x=-\infty, \Rightarrow r=-\infty$$

$$r=0 \Rightarrow x=-\sqrt{\frac{2E_b}{N_0}}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{2E_b/N_0}}e^{-x^2/2}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{\sqrt{2E_b/N_0}}^{\infty}e^{-x^2/2}dx=Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-x^2/2}dx$$

Assuming s_1 and s_2 are equiprobable

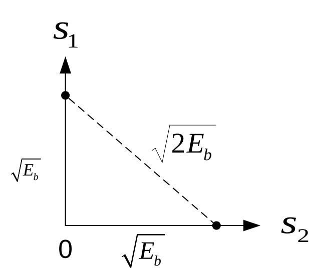
$$P_b=\frac{1}{2}P(e/s_1)+\frac{1}{2}P(e/s_2)=Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Note P_e depends on the ratio $\frac{E_b}{N_0}$

$$d_{12}=2\sqrt{E_b} \Rightarrow E_b=\frac{1}{4}d_{12}^{\;2}$$

$$\Rightarrow P_b=Q\left(\sqrt{\frac{d_{12}^{\;2}}{2N_0}}\right)$$

For Binary orthogonal signals



$$s_1 = [\sqrt{E_b} \quad 0] \qquad d_{12} = \sqrt{2E_b}$$
$$s_2 = [0 \quad \sqrt{E_b}]$$

If s_1 is transmitted

$$\bar{r} = [\sqrt{E_b} + n_1 \quad n_2]$$
$$C(\bar{r}, \bar{s}_m) = 2\bar{r}.\bar{s}_m - \|\bar{s}_m\|^2$$
$$C(\bar{r}, \bar{s}_2) = 2.[\sqrt{E_b} + n_1 \quad n_2].[0 \quad \sqrt{E_b}] - E_b \qquad -(1)$$
$$C(\bar{r}, \bar{s}_1) = 2.[\sqrt{E_b} + n_1 \quad n_2].[\sqrt{E_b} \quad 0] - E_b \qquad -(2)$$

(1) Can be further simplified to $2n_1\sqrt{E_b} - E_b$ and

(2) Can be further simplified to $2E_b + 2n_1\sqrt{E_b} - E_b = E_b + 2n_1\sqrt{E_b}$

Probability of error

$$C(\bar{r}, \bar{s}_2) > C(\bar{r}, \bar{s}_1)$$
$$P[e \mid \bar{s}_1] = P[C(\bar{r}, \bar{s}_2) > C(\bar{r}, \bar{s}_1)]$$

$$2E_b + 2n_1\sqrt{E_b} - E_b < 2n_2\sqrt{E_b} - E_b$$
$$E_b + (n_1 - n_2)\sqrt{E_b} < 0$$
$$n_2 - n_1 > \sqrt{E_b}$$
$$P(e \mid s_1) = P[n_2 - n_1 > \sqrt{E_b}]$$

n_1 and n_2 are zero mean Gaussian random variables

x= n_2-n_1 is zero mean Gaussian random variable with variance $= \sigma^2 = N_0$

$$P(n_2 - n_1 > \sqrt{E_b}) = \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{E_b}}^{\infty} e^{-x^2/2N_0} dx = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} e^{-x^2/2} dx = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Because of the symmetry s_2 is the same.

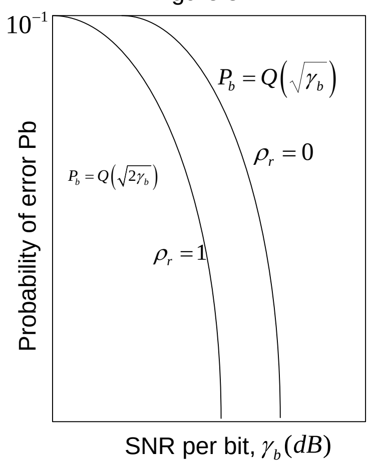
γ_b = SNR/bit

Compare orthogonal with antipodal (factor of 2 increase in energy)

$10\log_{10} 2 = -3dB \qquad d_{12}^2 = 2E_b \; for \quad orthogonal$

$\qquad d_{12}^2 = 4E_b \; for \quad antipodal$

Figure 5.2.4



Explain the concept of Union bound

In addition to the above antipodal and orthogonal examples, we can extend the analsys to other modulation techniques. For example,

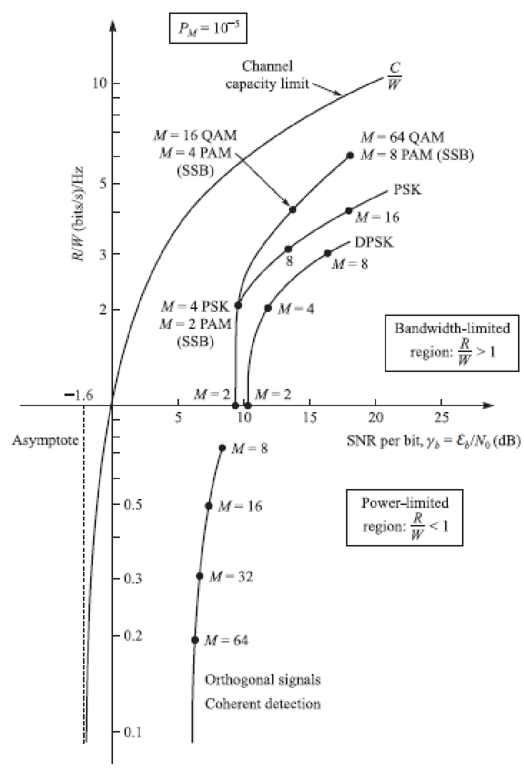
- Many orthogonal signals
- Bi-orthogonal
- Simplex
- M-ary PAM

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{d^2 E_g}{N_0}}\right)$$

$$= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M) E_{bav}}{(M^2-1) N_0}}\right)$$

Because $d^2 E_g = \frac{6}{M^2-1} P_{av} T$

- M-ary PSK
- QAM



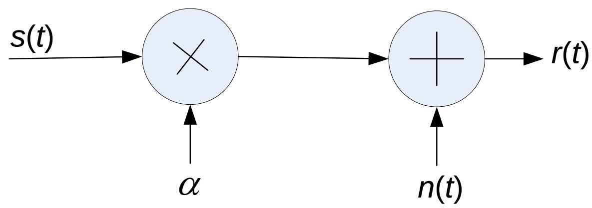
Comparison of Digital Modulation Methods

P 226-229 . Power spectral Efficiency!

Reading Material "Quiz"

Performance analysis for wire-line and radio communication Systems

Regenerative repeaters



Mathematical model for channel with attenuation and additive noise is

$$r(t) = \alpha s(t) + n(t)$$

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$ PAM Binary.

k→ Repeaters assuming single error at a time.

$$P \approx kQ\left(\sqrt{\frac{2E_b}{N_0}}\right)$$ at repeater….. Why not exact equal sign? (error cancellation)

Analog: Required E_b/N_0 reduced by k

$$P \approx Q\left(\sqrt{\frac{2E_b}{kN_0}}\right)$$

At decision receiver →note receiver is connected k= 1+ repeater.

Example

1000 km 10 km repeater k=100 10^{-5}

$$10^{-5} = 100\,Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \Rightarrow 10^{-7} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \Rightarrow \text{SNR} = 11.3\text{dB}$$

$$10^{-5} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \Rightarrow \text{SNR} = 29.6\text{ dB}$$ 29.6-11.3 = 18.3 dB

Link Budget analysis in radio comm. Systems

- Microwave line-of-sight Transmission
- Link budget analysis

$$P_R = \frac{P_T G_T A_R}{4\pi d^2} \qquad (1)$$

The design should specify P_T, size of antenna (transmit and receiver)

SNR required to achieve a given performance and data rate.

G_T: antenna transmit gain $\frac{P_T G_T}{4\pi d^2}$, G_T =1 for isotropic antenna.

$P_T G_T$: Effective radiated power.

ERP or *EIRP* compared with isotropic antenna

A_R: effective area of the antenna, $A_R = \frac{G_R \lambda^2}{4\pi}$ (2)

$c = \lambda f$, $c = 3\text{X}10^8$ m/s

Substitute (2) in (1)

$$P_R = \frac{P_T G_T G_R}{(4\pi d/\lambda)^2}$$

Free space path loss factor $L_s = \left(\frac{\lambda}{4\pi d}\right)^2$

Additional losses "atmospheric" L_a.

$$P_R = P_T G_T G_R L_s L_a$$

Calculation of the antenna gain is antenna specific and depends on (dimensions) diameter *D*,
Illumination efficiency factor

Effective area A_R

Area *A*

Beam width θ_B

Dish, horn ……etc

$(P_R)_{dB} = (P_T)_{dB} + (G_T)_{dB} + (G_R)_{dB} + (L_s)_{dB} + (L_o)_{dB}$

Example

A geosynchronous satellite orbit (36,000 km)

Power radiated is 100 W 20 dB above 1W -- 20dBW

Transmit antenna gain 17dB ERP = 37 dBW

3-m parabolic antenna at 4 GHz 'downlink'

η = 0.5 efficiency factor

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2 = 39dB$$

$$L_s = \left(\frac{\lambda}{4\pi d}\right)^2 = 195.6dB$$

$(P_R)_{dB}$ = 20 + 17 + 39 – 195.6 = -119.6 dBW

$P_R = 1.1 \text{ X } 10^{-12}$ W

is this low or high ?

What matter is the SNR.

Noise is flat for up to 10^{-12} Hz
$N_0 = k_B T_0$ W/Hz

K_B : is Boltzmann's constant $1.38X10^{-23}$
Total noise NW

Performance is dependent on $\frac{E_b}{N_0} = \frac{T_b P_R}{N_0} = \frac{1}{R}\frac{P_R}{N_0}$

$$\frac{P_R}{N_0} = R\left(\frac{E_b}{N_0}\right)_{Req}$$

Example for the same previous example.
P_R =1.1 X 10^{-12} W (-119.6 dBW)
N_0 = 4.1 X 10^{-21} W/Hz, $P_r = N_0 W = K_B T_0 W$
= -203.9 dBW/Hz

$\frac{P_R}{N_0}$ = -119.6 + 203.9 = 84.3 dB Hz

$\frac{E_b}{N_0}$ SNR is 10dB

R_{dB} = 84.3-10 = 74.3 dB with respect to 1bit/sec
= 26.9 Mbps
420 PCM (64000 bps)

The introduced safety margin

$$R_{dB} = \left(\frac{P_R}{N_0}\right) - \left(\frac{E_b}{N_0}\right)_{req} - M_{dB}$$

$$= (P_T)_{dBW} + (G_T)_{dB} + (G_R)_{dB} + (L_a)_{dB} + (L_s)_{dB} - \left(\frac{E_b}{N_0}\right)_{req} - M_{dB}$$