Solution of Electromagnetic Field Theory

Subject Code: KEE 301

for B.Tech 3rd Sem (EE)

SECTION A

a) $\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$

$= 10\sin^2\theta\cos\phi\,\hat{a}_r + 10\sin 2\theta\cos\phi\,\hat{a}_\theta - 10\sin\theta\sin\phi\,\hat{a}_\phi$

b) Differential surfaces in cylindrical

1) $d\vec{S}_1 = \rho\, d\rho\, d\phi\, \hat{a}_z$

$d\vec{S}_2 = \rho\, d\phi\, dz\, \hat{a}_\rho$

$d\vec{S}_3 = d\rho\, dz\, \hat{a}_\phi$

Differential Volume

$dv = \rho\, d\rho\, d\phi\, dz.$

Differential surfaces in spherical surfaces

$d\vec{S}_1 = r^2\sin\theta\, d\theta\, d\phi\, \hat{a}_r$

$d\vec{S}_2 = r\sin\theta\, dr\, d\phi\, \hat{a}_\theta$

$d\vec{S}_3 = r\, dr\, d\theta\, \hat{a}_\phi$

Differential Volume

$dv = r^2\sin\theta\, dr\, d\theta\, d\phi$

c) Component of $\vec{A}$ along $\vec{B} = (\vec{A}\cdot\hat{a}_B)\hat{a}_B$

d) Stoke's theorem: $\oint \vec{A}\cdot d\vec{l} = \int_S (\nabla\times\vec{A})\cdot d\vec{S}$

Gauss's Law : Total electric flux Ψ through any closed surface is equal to the total charge enclosed by that surface

$Q_{enc} = \Psi$ or $\int_v \rho_v\, dv = \oint_S \vec{D}\cdot d\vec{S}$

e) $r = \sqrt{x^2+y^2+z^2} = \sqrt{5^2+6^2+3^2} = 8.36$

$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{\sqrt{5^2+6^2}}{3} = 87.18^\circ$

$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{6}{5}\right) = 50.19^\circ$

SECTION B

2 (A)
$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$B_\rho = \frac{10}{r}\sin\theta + r\cos^2\theta$

$B_\phi = 1$

$B_z = \frac{10}{r}\cos\theta - r\sin\theta\cos\theta$

At $(5, \frac{\pi}{2}, -2)$

$\vec{B} = 2.467\,\hat{a}_\rho + \hat{a}_\phi + 1.167\,\hat{a}_z$

2(B) i) $dS = \int_{\phi=0}^{\pi/4}\int_{\rho=1}^{3} \rho\, d\rho\, d\phi = 3.142$

ii) $dS = \int_{\phi=0}^{3\pi}\int_{\theta=\pi/4}^{2\pi/3} r^2\sin\theta\, d\theta\, d\phi \Big|_{r=10} = 758.4$

2(C) $Q_1 = 2\,mC$, $Q_2 = -3\,mC$, $Q_3 = 5\,mC$

$\vec{R}_1 = 4\hat{a}_z$, $\vec{R}_2 = -2\hat{a}_x + 5\hat{a}_y + \hat{a}_z$, $\vec{R}_3 = 3\hat{a}_x - 4\hat{a}_y + 6\hat{a}_z$

$\vec{R} = -\hat{a}_x + \hat{a}_y + 2\hat{a}_z$

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0|\vec{R}-\vec{R}_1|} + \frac{Q_2}{4\pi\epsilon_0|\vec{R}-\vec{R}_2|} + \frac{Q_3}{4\pi\epsilon_0|\vec{R}-\vec{R}_3|}$$

2(d) $\vec{M} = 2\hat{a}_x - 6\hat{a}_y + 5\hat{a}_z$ $\quad \vec{N} = 3\hat{a}_y + \hat{a}_z$

$\vec{M}\cdot\vec{N} = 2\cdot 0 + (-6)\cdot 3 + 5\cdot 1 = -18 + 5 = -13$

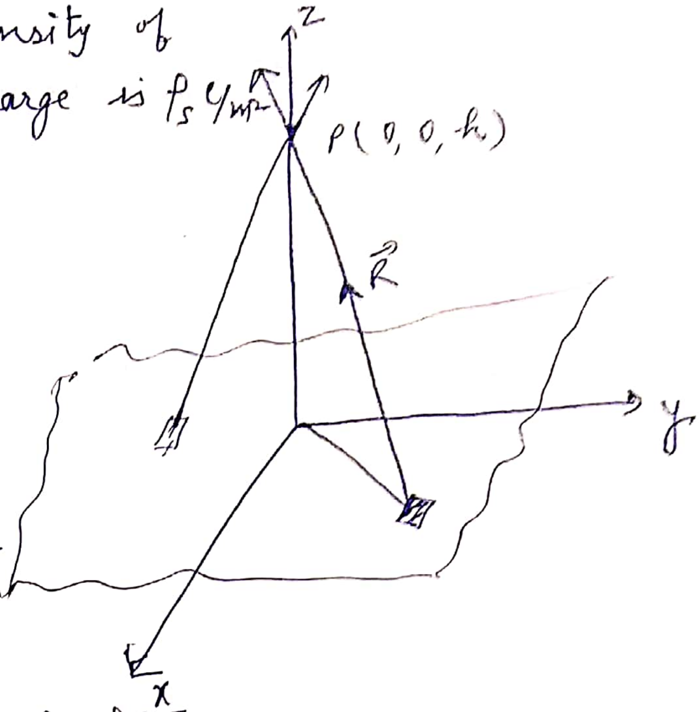
$$\vec{M}\times\vec{N} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -6 & 5 \\ 0 & 3 & 1 \end{vmatrix} = (-6-15)\hat{a}_x + (0-2)\hat{a}_y + (6-0)\hat{a}_z$$
$$= -21\hat{a}_x - 2\hat{a}_y + 6\hat{a}_z$$

$\cos\theta = \frac{\vec{M}\cdot\vec{N}}{|M|\,|N|}$

$\therefore \theta = \cos^{-1}\left(\frac{\vec{M}\cdot\vec{N}}{|M|\cdot|N|}\right)$

SECTION C

3(a) Let charge density of infinite sheet of charge is ρ_s C/m²



$\therefore dQ = \rho_s dS$

$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\hat{a}_R$$

$$\vec{R} = -\rho\hat{a}_\rho + h\hat{a}_z$$

$$R = [\rho^2 + h^2]^{1/2}$$

$$\hat{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{-\rho\hat{a}_\rho + h\hat{a}_z}{[\rho^2+h^2]^{1/2}}$$

$$dQ = \rho_s dS = \rho_s \rho\, d\phi\, d\rho$$

$$\therefore d\vec{E} = \frac{\rho_s \rho\, d\phi\, d\rho\,[-\rho\hat{a}_\rho + h\hat{a}_z]}{4\pi\epsilon_0[\rho^2+h^2]^{3/2}}$$

$$\vec{E} = \frac{\rho_s}{4\pi\epsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\rho\, d\phi}{[\rho^2+h^2]^{3/2}}\hat{a}_z$$

$$\boxed{\vec{E} = \frac{\rho_s}{2\epsilon_0}\hat{a}_z}$$

3(b) $\vec{A} = xy\,\hat{a}_x + y^2\hat{a}_y + xz\,\hat{a}_z$

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

$$= y + 2y + x$$

$$= x + 3y$$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\hat{a}_x + \left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right)\hat{a}_y + \left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)\hat{a}_z$$

$$= (0-0)\hat{a}_x + (0 - z)\hat{a}_y + (2y - x)\hat{a}_z$$

$$= -z\,\hat{a}_y + (2y - x)\hat{a}_z.$$

4(a) $\oint_L \vec{A}\cdot d\vec{l} = \int_a^b \vec{A}\cdot d\vec{l} + \int_b^c \vec{A}\cdot d\vec{l} + \int_c^d \vec{A}\cdot d\vec{l} + \int_d^a \vec{A}\cdot d\vec{l}$

$$\int_a^b \vec{A}\cdot d\vec{l} = \int_{\phi=60^\circ}^{30^\circ} \rho \sin\phi \, d\phi = 2(-\cos\phi)\Big|_{60^\circ}^{30^\circ} = -(\sqrt{3}-1)$$

$$\int_b^c \vec{A}\cdot d\vec{l} = \int_{\rho=2}^{5} \rho \cos\phi \, d\rho = \cos 30^\circ \frac{\rho^2}{2}\Big|_2^5 = \frac{21\sqrt{3}}{4}$$

$$\int_c^d \vec{A}\cdot d\vec{l} = \int_{\phi=30^\circ}^{60^\circ} \rho \sin\phi \, d\phi = 5(-\cos\phi)\Big|_{30^\circ}^{60^\circ} = \frac{5}{2}(\sqrt{3}-1)$$

$$\int_d^a \vec{A}\cdot d\vec{l} = \int_{\rho=5}^{2} \rho \cos\phi \, d\rho = \cos 60^\circ \frac{\rho^2}{2}\Big|_5^2 = -\frac{21}{4}$$

$$\therefore \oint \vec{A}\cdot d\vec{l} = -\sqrt{3}+1+\frac{21\sqrt{3}}{4}+\frac{5\sqrt{3}}{2}-\frac{5}{2}-\frac{21}{4} = \frac{27}{4}(\sqrt{3}-1)$$

4(b) **Transformation matrices**

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}.$$

5 (a) $\vec{P} = (2y^2 + z)\hat{a}_x + 4xy\,\hat{a}_y + x\,\hat{a}_z$ C/m²

volume charge density

$$\rho_v = \nabla\cdot\bar{D} = \nabla\cdot\vec{P}$$

$$= \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z$$

$$= 0 + 4x + 0$$

$$= 4x$$

ρ_v at $(-1, 0, 3) = 4(-1) = -4$ C/m³.

5 (b) $W_E = W_1 + W_2 + W_3 + W_4 = 0 + Q_2V_{21} + Q_3(V_{31} + V_{32}) + Q_4(V_{41} + V_{42} + V_{43})$

If the charges are positioned in reverse order

$$W_E = W_4 + W_3 + W_2 + W_1$$

$$= 0 + Q_3V_{34} + Q_2(V_{24} + V_{23}) + Q_1(V_{14} + V_{13} + V_{12})$$

$$2W_E = Q_1(V_{12} + V_{13} + V_{14}) + Q_2(V_{21} + V_{23} + V_{24}) + Q_3(V_{31} + V_{32} + V_{34}) + Q_4(V_{41} + V_{42} + V_{43})$$

$$\boxed{W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3 + Q_4V_4)}$$

Potentials due to continuous charge distributions :-

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int_L \frac{\rho_L(\bar{r}')\,dl'}{|\bar{r} - \bar{r}'|}$$ (due to line charge)

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s(\bar{r}')\,ds'}{|\bar{r} - \bar{r}'|}$$ (due to surface charge)

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_v(\bar{r}')\,dv'}{|\bar{r} - \bar{r}'|}$$ (due to volume charge)