The auxiliary equation is $5s^2 + 5 + K = 0$

$$5s^2 + 45 = 0$$

$\Rightarrow$ $s = \pm j3.$

8. **Angle of departure:**

$$\phi_{d1-(-2+j)} = 180 - (\phi_{p3} + \phi_{p4})$$

$$= 180 - (135 + 90) = -45$$

$$\phi_{d2-(-2-j)} = +45\,.$$

The complete root locus plot for the above-mentioned system is shown in Figure (g).

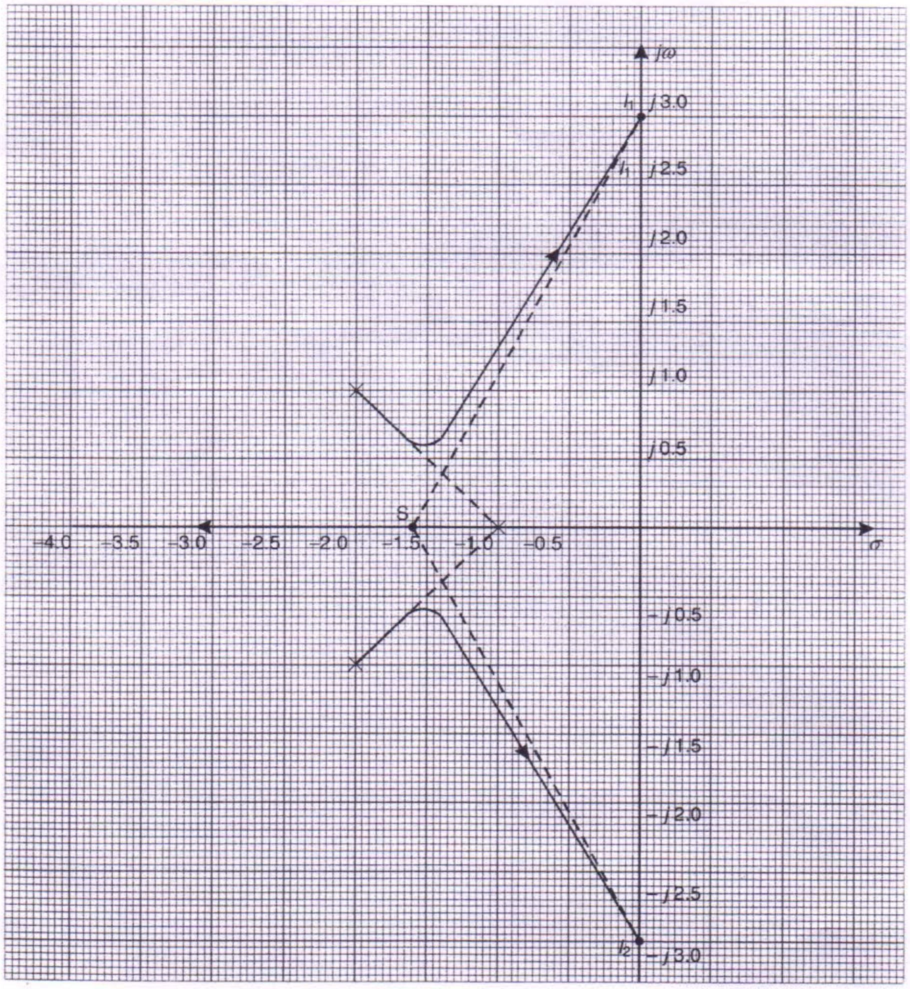

Figure (g) Root locus plot for Problem 10.11

ADDITIONAL SOLVED PROBLEMS

ASP-1: Calculate the angle of asymptotes and the centroid for the system having $G(s)H(s) = \dfrac{K(s+3)}{s(s+2)(s+4)(s+5)}$.

Solution:

Given that

$$G(s)H(s) = \frac{K(s+3)}{s(s+2)(s+4)(s+5)}.$$

The poles are given as

$$s(s+2)(s+5)(s+4) = 0$$

$$s = 0,\ s = -2,\ s = -5,\ s = -4.$$

The zeros are given as

$$s = -3$$

Number of poles = $P = 4$.
Number of zeros = $Z = 1$.
Number of root locus branches = $N = P = 4$.
Number of asymptotic lines = $n = P - Z = 3$.

1. The angle of asymptotic lines with negative real axis

$$\phi = \frac{(2q+1)180}{P-Z},\ \text{where } q = 0, 1, 2$$

$$= 1 \times 180/3,\ 3 \times 180/3,\ 5 \times 180/3$$

$$= 60,\ 180,\ 300.$$

2. The centroid is given as

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z} = \frac{(0-2-4-5)-(-3)}{4-1} = -2.667.$$

ASP-2: Sketch the root locus plot of the system whose OLTF is given as $G(s)H(s) = \dfrac{K}{s(s^2+8s+32)}$.

Solution:

Given that

$$G(s)H(s) = \frac{K}{s(s^2+8s+32)}.$$

1. Root locus is symmetrical about real axis.
2. The root locus plot starts from a pole at $K = 0$ and terminates at an open-loop zero, that is, at $K = \infty$.
3. The poles are given as
   $s = 0,\ s^2 + 8s + 32 = 0$
   $P_1 = 0,\ P_2 = -4 + j4,\ P_3 = -4 - j4$
   Number of poles = $P = 3$
   Number of zeros = $Z = 0$

Number of root locus branches = $N = P = 3$
Number of asymptotic lines = $n = P - Z = 3$.

4. The angle of asymptotic lines with negative real axis

$$\phi = \frac{(2q+1)180}{P-Z}, \text{ where } q = 0, 1, 2$$
$$= 180/3,\ 3 \times 180/3,\ 5 \times 180/3$$
$$= 60,\ 180,\ 300.$$

5. The centroid is given as

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z} = \frac{(0-4+j4-4-j4)-0}{3-0} = -\frac{8}{3} = -2.667.$$

6. **Break-away points:**
The C.E. is given as

$$1 + G(s)H(s) = 0$$
$$1 + \frac{K}{s(s^2+8s+32)} = 0$$
$$K = -s(s^2 + 8s + 32)$$
$$K = -(s^3 + 8s^2 + 32s)$$
$$\frac{dK}{ds} = 0 \Rightarrow 3s^2 + 16s + 32 = 0$$
$$\Rightarrow \quad s_1 = 2.667 + j1.88,\ s_2 = -2.667 - j1.88.$$

The points are not on the root locus. Therefore, there are no break-away points.

7. **Intersection points of the root locus branches with imaginary axis:**
The C.E. is given as

$$1 + G(s)H(s) = 0$$
$$s^3 + 8s^2 + 32s + K = 0$$

s^3	1	32
s^2	8	K
s^1	$(256 - K)/8$	0
s^0	K	0

For a stable system, $K > 0$, $(256 - K)/8 > 0$

$$\Rightarrow \quad K > 0,\ K < 256$$

For a stable system, the maximum value of K is 256. For $K > 256$, the roots lie on the RHS of the s-plane, and hence, $K = 256$ is the value for marginal stability.
The auxiliary equation is

$$8s^2 + K = 0$$
$$8s^2 + 256 = 0$$
$$\Rightarrow \quad s = \pm j\,5.656.$$

8. **Angle of departure:**

$$\phi_{d1-(-4+j4)} = 180 - (\phi_{p3})$$

$$= 180 - (225) = -45$$

$$\phi_{d2-(-4-j4)} = +45.$$

The complete root locus plot for the above-mentioned system is as shown in Figure (h).

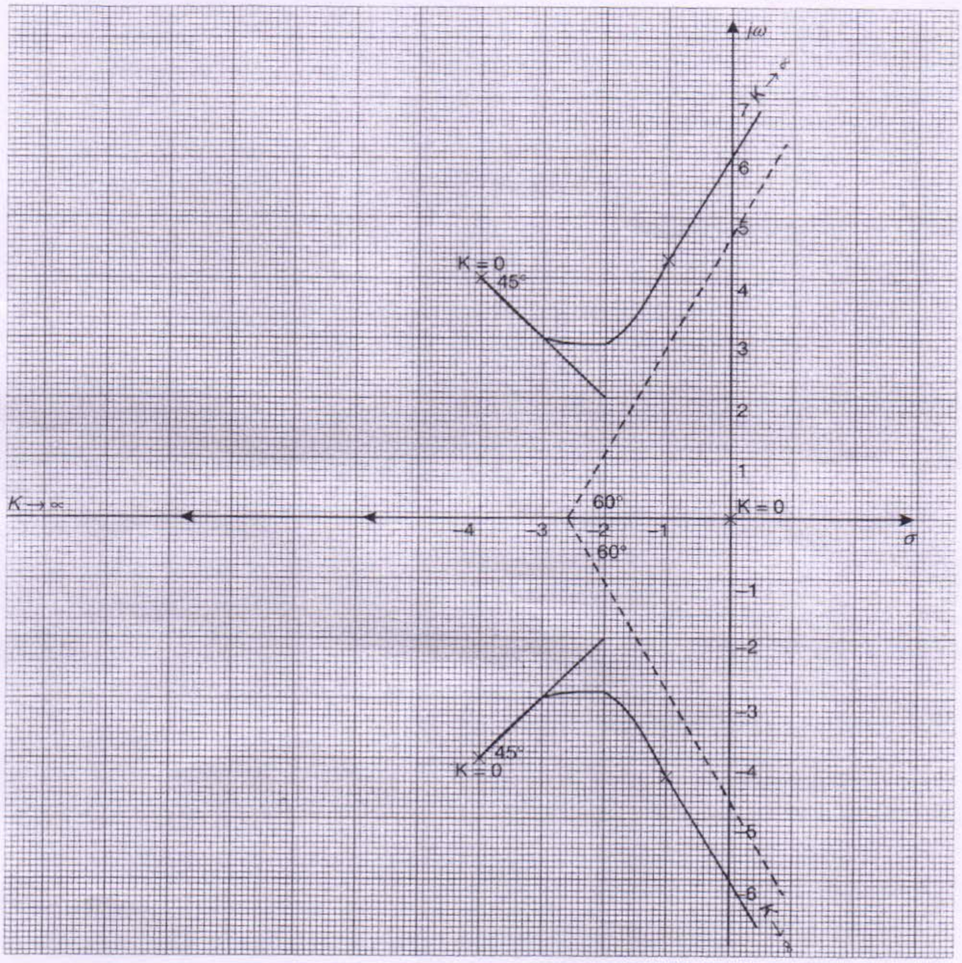

Figure (h) Root locus plot for ASP-2

ASP-3: Sketch the root locus plot of the system whose OLTF is given as $G(s)H(s) = \dfrac{K}{s(s+2)(s^2+2s+2)}$.

Solution:

Given that

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+2s+2)}.$$

1. Root locus is symmetrical about the real axis.
2. The root locus plot starts from a pole at $K = 0$ and terminates at an open-loop zero, that is, at $k = \infty$.

3. The poles are given as
   $s = 0, s = -2, s^2 + 2s + 2 = 0$
   $P_1 = 0, P_2 = -2, P_3 = -1 + j1, P_4 = -1 - j1$
   Number of poles = $P = 4$
   Number of zeros = $Z = 0$
   Number of root locus branches = $N = P = 4$
   Number of asymptotic lines = $n = P - Z = 4$
4. The angle of asymptotic lines with negative real axis

$$\phi = \frac{(2q+1)180}{P-Z}, \text{ where } q = 0, 1, 2, 3$$
$$= 180/4, 3 \times 180/4, 5 \times 180/4, 7 \times 180/4$$
$$= 45, 135, 225, 315.$$

5. The centroid is given as

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z} = \frac{(0-2-1+j1-1-j1)-(0)}{4-0} = -1.$$

6. **Break-away points:**
   The C.E. is given as

$$1 + G(s)H(s) = 0$$
$$1 + \frac{K}{s(s+2)(s^2+2s+2)} = 0$$
$$K = -s(s+2)(s^2+2s+2)$$
$$K = -s(s^4 + 4s^3 + 6s^2 + 4s)$$
$$\frac{dK}{ds} = 0 \Rightarrow 4s^3 + 12s^2 + 12s + 4 = 0$$
$$\Rightarrow \quad s^3 + 3s^2 + 3s + 1 = 0$$
$$\Rightarrow \quad (s+1)^3 = 0 \Rightarrow s = -1$$

   The valid break-away point is $B_1 = -1$.
7. **Intersection points of the root locus branches with imaginary axis:**
   The C.E. is given as

$$1 + G(s)H(s) = 0$$
$$s^4 + 4s^3 + 6s^2 + 4s + K = 0$$

s^4	1	6	K
s^3	4	4	0
s^2	5	K	0
s^1	$(20-4K)/5$	0	0
s^0	K	0	0

For a stable system, $K > 0, (20 - 4K)/4 > 0$
$\Rightarrow$ $K > 0, K < 5$

For a stable system, the maximum value of K is 5. For $K > 5$, the roots lie on the RHS of the s-plane, and hence, $K = 5$ is the value for marginal stability.

The auxiliary equation is

$$5s^2 + K = 0$$

$$\Rightarrow \quad 5s^2 + 5 = 0$$

$$\Rightarrow \quad s = \pm j1$$

8. **Angle of departure:**

$$\phi_{d1-(-1+j1)} = 180 - (\phi_{p1} + \phi_{p2} + \phi_{p4})$$

$$= 180 - (135 + 45 + 90) = -90$$

$$\phi_{d2-(-1-j1)} = +90$$

The complete root locus plot is as shown in Figure (i).

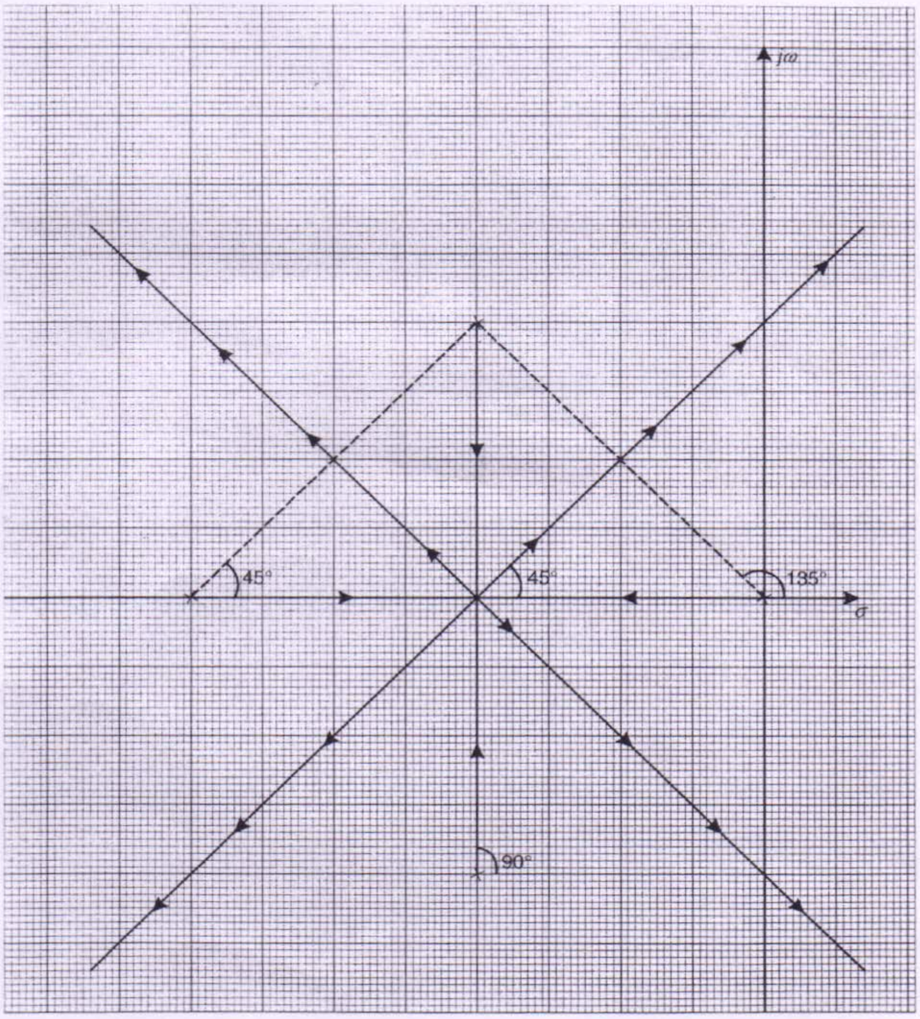

Figure (i) Root locus plot for ASP-3

ASP-4: Sketch the root locus plot for a unity feedback system whose OLTF is given as $G(s) = \dfrac{K(s+0.5)}{s^2(s+4.5)}$.

Solution:

Given that

$$G(s) = \frac{K(s+0.5)}{s^2(s+4.5)}.$$

1. Root locus is symmetrical about real axis.
2. The root locus plot starts from a pole at $K = 0$ and terminates at an open-loop zero, i.e., at $K = \infty$.
3. The poles are given as
   $s = 0, s = 0, s = -4.5$
   $P_1 = 0, P_2 = 0, P_3 = -4.5$
   The zeros are given as
   $s = -0.5$
   Number of poles $= P = 3$
   Number of zeros $= Z = 1$
   Number of root locus branches $= N = P = 3$
   Number of asymptotic lines $= n = P - Z = 2$
4. The angle of asymptotic lines with negative real axis
   $$\phi = \frac{(2q+1)180}{P-Z}, \text{ where } q = 0, 1$$
   $$= 180/2, 3 \times 180/2$$
   $$= 90, 270.$$
5. The centroid is given as
   $$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z} = \frac{-4.5-(-0.5)}{2} = -2.$$
6. **Break-away points:**
   The C.E. is given as
   $$1 + G(s)H(s) = 0$$
   $$1 + \frac{K(s+0.5)}{s^2(s+4.5)} = 0$$
   $$\Rightarrow \quad K = -\frac{s^2(s+4.5)}{s+0.5}$$
   $$\frac{dK}{ds} = 0 \Rightarrow 2s^3 + 6s^2 + 4.5s = 0$$
   $$\Rightarrow \quad s(s^2 + 3s + 2.25) = 0$$
   $$\Rightarrow \quad s(s + 1.5)^2 = 0$$
   $$\Rightarrow \quad s_1 = 0, s_2 = -1.5, s_3 = -1.5.$$
   The valid break-away points are $B_1 = 0, B_2 = -1.5$.
7. **Angle of departure:**
   It is not necessary to find out the angle of departure.
8. **Intersection points of the root locus branches with imaginary axis:**
   The C.E. is given as
   $$1 + G(s)H(s) = 0$$

$$1+\frac{k(s+0.5)}{s^2(s+4.5)}=0$$

$\Rightarrow$ $$s^3+4.5s^2+Ks+0.5K=0$$

s^3	1	K	0
s^2	4.5	$0.5K$	0
s^1	$0.88K$	0	0
s^0	$0.5K$	0	0

For a stable system, $0.88K > 0,\ 0.5K > 0$

$\Rightarrow$ $K > 0.$

The range of values of K is $0 < K < \infty$.

The complete root locus plot is as shown in Figure (j).

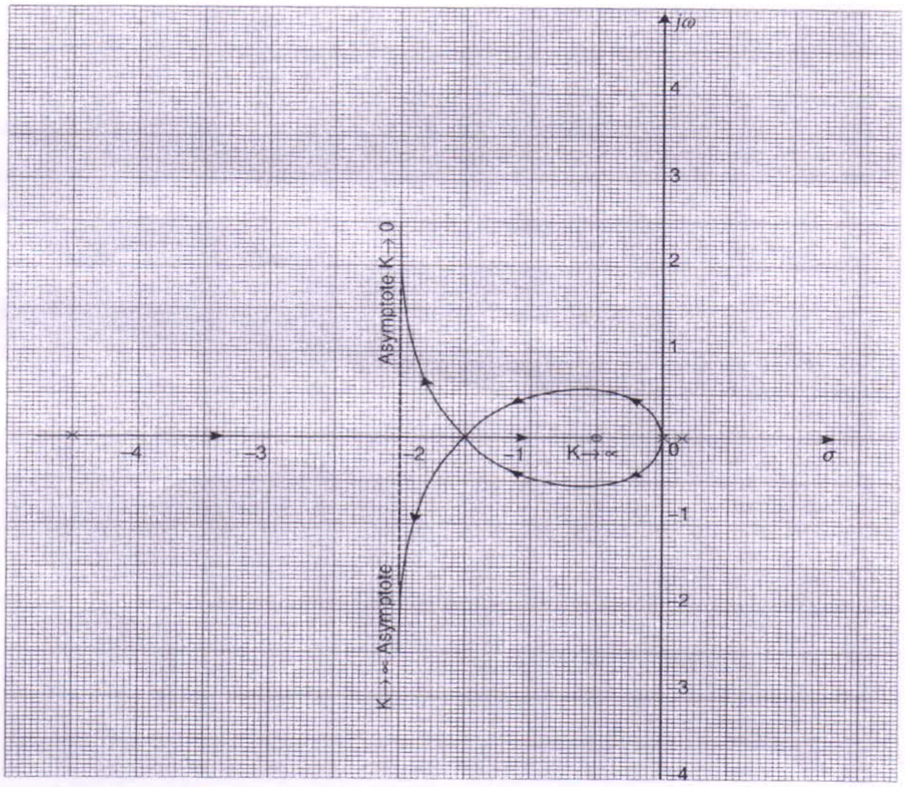

Figure (j) Root locus plot for ASP-4

ASP-5: Sketch the root locus plot for a unity feedback system whose OLTF is given as $G(s)=\frac{K(s+1)}{(s-1)(s+2)(s+4)}$. Find the range of K for which the system is stable.

Solution:

Given that

$$G(s) = \frac{K(s+1)}{(s-1)(s+2)(s+4)}.$$

1. Root locus is symmetrical about the real axis.
2. The root locus plot starts from a pole at $K = 0$ and terminates at an open-loop zero, that is, at $K = \infty$.
3. The poles are given as
   $s = 1, s = -2, s = -4$
   $s = -1$
   Number of poles $= P = 3$
   Number of zeros $= Z = 1$
   Number of root locus branches $= N = P = 3$
   Number of asymptotic lines $= n = P - Z = 2$
4. The angle of asymptotic lines with negative real axis
   $$\phi = \frac{(2q+1)180}{P-Z}, \text{ where } q = 0, 1$$
   $$= 180/2,\ 3 \times 180/2$$
   $$= 90,\ 270$$
5. The centroid is given as
   $$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z} = \frac{(1-2-4)-(-1)}{2} = -2.$$
6. **Break-away points:**
   The C.E. of the system is given as
   $$1 + G(s)H(s) = 0$$
   $$1 + \frac{K(s+1)}{(s-1)(s+2)(s+4)} = 0.$$
   $$K = -\frac{(s^3+5s^2+2s-8)}{s+1}$$
   $$\frac{dK}{ds} = 0 \Rightarrow 2s^3 + 8s^2 + 10s + 10 = 0 \Rightarrow s^3 + 4s^2 + 5s + 5 = 0.$$
   After solving the above-mentioned equation, the valid break-away points is $B_1 = -2.863$
7. **Angle of departure:**
   It is not necessary to find out the angle of departure.
8. **Intersection points of the root locus branches with imaginary axis:**
   The C.E. of the system is given as
   $$1 + G(s)H(s) = 0$$
   $$1 + \frac{K(s+1)}{(s-1)(s+2)(s+4)} = 0$$
   $$s^3 + 5s^2 + 2s - 8 = -K(s+1)$$
   $$s^3 + 5s^2 + (2+K)s - 8 + K = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 2+K \\ s^2 & 5 & -8+K \\ s^1 & \dfrac{4K+18}{5} & 0 \\ s^0 & -8+K & 0 \end{array}$$

For a stable system, $-8 + K > 0$, $2 + 6K > 0$

$\Rightarrow \qquad K > 8,\ K > -1/3$

The range of K for the system being stable is $8 < K < \infty$. The asymptotes are 90° and 270° so that the locus do not cross the $j\omega$-axis.

The complete root locus plot is as shown in Figure (k).

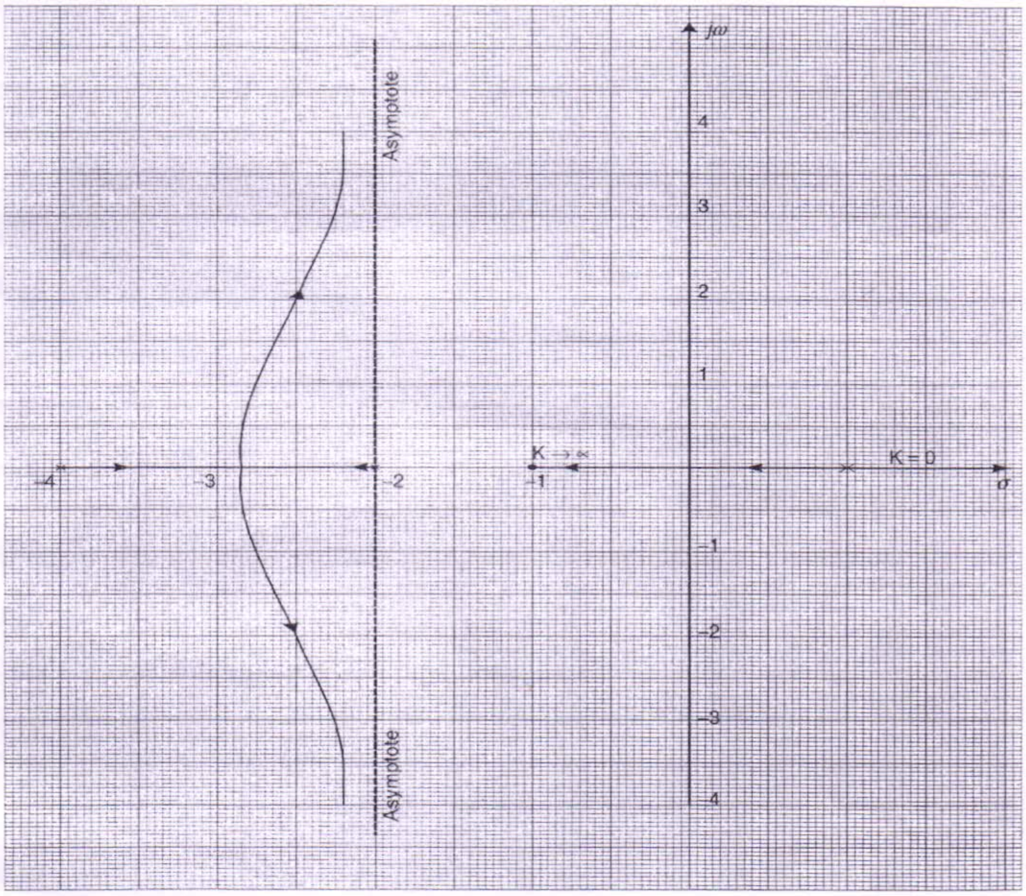

Figure (k) Root locus plot for ASP-5

ASP-6: Sketch the root locus plot of a unity feedback system whose OLTF is $G(s) = \dfrac{K(s+9)}{s(s^2+4s+11)}$.

Solution:

Given that

$$G(s) = \frac{K(s+9)}{s(s^2+4s+11)}.$$

1. Root locus is symmetrical about real axis.
2. The root locus plot starts from a pole at $K = 0$ and terminates at an open-loop zero, that is, at $K = \infty$.
3. The poles are given as

   $S = 0, s = -2 + j2.64, s = -2 - j2.64$

   The zeros are given as

   $s = -9$

   Number of poles $= P = 3$

   Number of zeros $= Z = 1$

   Number of root locus branches $= N = P = 3$

   Number of asymptotic lines $= n = P - Z = 2$
4. The angle of asymptotic lines with negative real axis

$$\phi = \frac{(2q+1)180}{P-Z}, \text{ where } q = 0, 1$$

$$= 180/2, 3 \times 180/2$$

$$= 90, 270.$$

5. The centroid is given as

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{P-Z} = \frac{(0-2+j2.64-2-j2.64)-(-9)}{3-1} = 2.5.$$

6. **Break-away points:**

   The C.E. of the system is given as

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K(s+9)}{s(s^2+4s+11)} = 0$$

$$K = \frac{s(s^2+4s+11)}{s+9}$$

$$\frac{dK}{ds} = 0 \Rightarrow 2s^3 + 31s^2 + 61s = 0$$

$$\Rightarrow \quad s(s^2 + 15.5s + 30.5) = 0$$

$$\Rightarrow \quad s = 0, s = -13.157, s = -2.313.$$

   There are no valid break-away points.

7. **Angle of departure:**

$$\phi_{d1-(-2+j2.64)} = 180 - (\phi_{p1} + \phi_{p3}) + \phi_{z1}$$

$$= 180 - (127.1 + 90)20.7 = 16.4$$

$$\phi_{d2-(-2-j2.64)} = +16.4.$$

8. **Intersection points of the root locus branches with imaginary axis:**

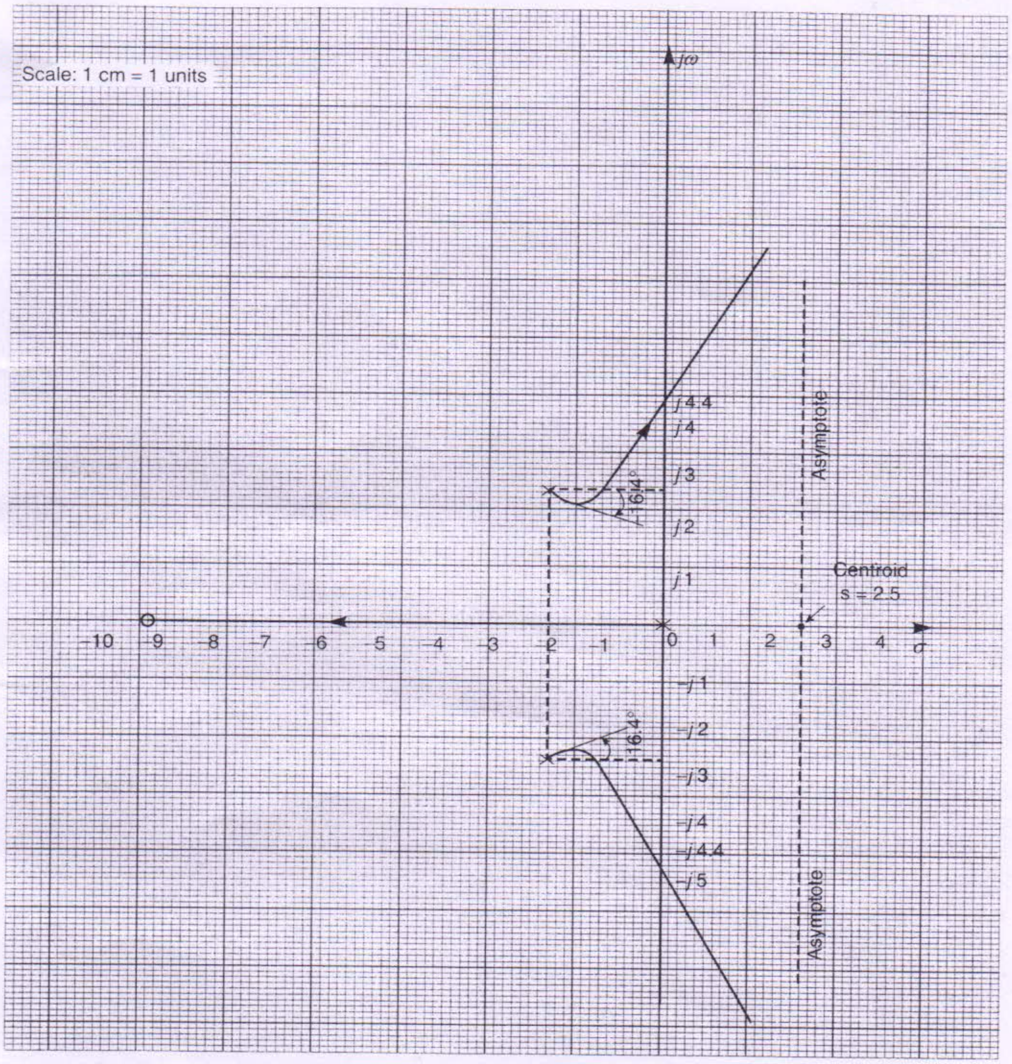

Figure (l) Root locus plot for ASP-6

The C.E. of the system is given as

$$1+G(s)H(s)=0$$

$$1+\frac{K(s+9)}{s(s^2+4s+11)}=0$$

$$\Rightarrow \qquad s(s^2 + 4s + 11) + K(s + 9) = 0$$

$$\Rightarrow \qquad s^3 + 4s^2 + (11 + K)s + 9K = 0$$

s^3	1	$11+K$	0
s^2	4	$9K$	0
s^1	$(44-5K)/4$	0	0
s^0	$9K$	0	0

For a stable system, $9K > 0,\ (44 - 5K/4) > 0$

$\Rightarrow \qquad K > 0,\ K < 8.8.$

For a stable system, the maximum value of K is 8.8. For $K > 8.8$, the roots lie on the RHS of the s-plane, and hence, $K = 8.8$ is the value for marginal stability.

The auxiliary equation is $4s^2 + 9K = 0$.

$$\Rightarrow \qquad 4s^2 + 79.2 = 0$$

$$\Rightarrow \qquad s = \pm j\,4.45$$

The complete root locus plot is as shown in Figure (l).

ASP-7: Sketch the root locus plot of a unity feedback system whose OLTF is $G(s) = \dfrac{K(s+1.5)}{s(s+1)(s+5)}$.

***Solution*:**

Given that

$$G(s) = \frac{K\,(s+1.5)}{s(s+1)(s+5)}.$$

1. Root locus is symmetrical about real axis.
2. The root locus plot starts from a pole at $K = 0$ and terminates at an open-loop zero, i.e., at $K = \infty$.
3. The poles are given as
   $s = 0,\ s = -1,\ s = -5.$
   The zeros are given as
   $s = -1.5$
   Number of poles = $P = 3$
   Number of zeros = $Z = 1$
   Number of root locus branches = $N = P = 3$
   Number of asymptotic lines = $n = P - Z = 2$
4. The angle of asymptotic lines with negative real axis

$$\phi = \frac{(2q+1)180}{P-Z}, \text{ where } q = 0,\ 1$$

$$= 180/2,\ 3 \times 180/2$$

$$= 90,\ 270.$$